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OF MATHEMATICAL SCIENCE

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ON A SMALL VARIATION WHICH RENDERS A LINEAR DIFFERENTIAL SYSTEM INCOMPATIBLE.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society, April 25, 1914.)

LET us consider a homogeneous linear differential expression of the nth order*

$$L(u) \equiv l_n \frac{d^n u}{dx^n} + l_{n-1} \frac{d^{n-1} u}{dx^{n-1}} + \cdots + l_1 \frac{du}{dx} + l_0 u,$$

whose coefficients are continuous functions of the real variable x in a closed interval ab. We suppose that l_n does not vanish in this interval. We consider the $2n$ quantities

$$u(a),\ u'(a),\ \cdots,\ u^{[n-1]}(a); \qquad u(b),\ u'(b),\ \cdots,\ u^{[n-1]}(b)$$

and form n linearly independent linear forms in them, $U_1(u)$, $\cdots$, $U_n(u)$, with constant coefficients.

Consider now the homogeneous linear differential system

$$(1) \qquad L(u) = 0, \qquad U_i(u) = 0 \quad (i = 1, 2, \cdots, n).$$

This system is said to have k-fold compatibility if there are k and only k linearly independent functions which satisfy it. It is well known and immediately obvious that, if y_1, $\cdots$, y_n is any fundamental system of the equation $L(u) = 0$, a necessary and sufficient condition for k-fold compatibility is that the rank of the matrix

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* No additional difficulties would be introduced if we considered the more general expressions treated in my paper, *Transactions*, vol. 14 (1913), p. 403. See in particular the latter part of § 3.

$$(2)\qquad \begin{pmatrix} U_1(y_1) & \cdots & U_1(y_n) \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ U_n(y_1) & \cdots & U_n(y_n) \end{pmatrix}$$

be $n - k$. Since all the elements, and hence all the determinants, of this matrix will be only slightly changed by small variation of the coefficients of the system (1) (provide that, as is obviously possible, the y_i's and their first $n -$ derivatives are allowed to vary only slightly) we immediately infer the following important result:

THEOREM I. *If the system* (1) *has k-fold compatibility, it has no higher order of compatibility after any variation of its coefficients which is uniformly sufficiently small in ab.*[*]

While this theorem tells us that *no* very small variation will *raise* the order of compatibility, the main result to be established in this paper refers to the possibility of *lowering* the order of compatibility, and here we shall prove not merely that there always exist arbitrarily small variations which render the system incompatible (i. e., reduce its order of compatibility to zero) but that a variation of a very simple and important type will have this effect; namely a *real* variation of the coefficient l_0 alone (so that the conditions $U_i = 0$ are not varied) and, indeed, a variation which is everywhere positive, or, what is not essentially different, everywhere negative. The proof will depend on certain preliminary lemmas.

Let us suppose that the system (1) has k-fold compatibility, and, as a matter of notation, let us suppose that the $(n - k)$-rowed determinant in the upper left-hand corner of the matrix (2) is not zero. Then every solution of the equation $L(u) = 0$ which satisfies the first $n - k$ conditions $U_i = 0$ will also satisfy the remaining conditions. Such a function is given by the determinant

$$(3)\qquad \begin{vmatrix} y_1 & \cdots & y_{n-k} & c_1 y_{n-k+1} + \cdots + c_k y_n \\ U_1(y_1) & \cdots & U_1(y_{n-k}) & c_1 U_1(y_{n-k+1}) + \cdots + c_k U_1(y_n) \\ \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ U_{n-k}(y_1) & \cdots & U_{n-k}(y_{n-k}) & c_1 U_{n-k}(y_{n-k+1}) + \cdots + c_k U_{n-k}(y_n) \end{vmatrix}$$

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[*] The special case $k = 0$ of this theorem tells us that if the system (1) is incompatible, it remains so after every variation of its coefficients which is uniformly sufficiently small.

Moreover this determinant vanishes identically only when $c_1, \cdots, c_k$ are all zero, since otherwise $y_1, \cdots, y_n$ would be linearly dependent. Consequently the formula (3) gives a linear family of solutions of the system (1) whose bases consist of just k functions, so that, since by hypothesis (1) has k-fold compatibility, (3) gives its general solution.

Let us now suppose that the coefficients of $L(u)$ are continuous functions of (x, λ) and that the coefficients of the U_i's are continuous functions of λ; and that when $\lambda = \lambda_0$ and for a certain neighborhood of this value the system (1) has just k-fold compatibility. If we arrange the notation so that when $\lambda = \lambda_0$ the $(n - k)$-rowed determinant in the upper left-hand corner of (2) is not zero, we can take the neighborhood of λ_0 so small that this same determinant does not vanish in this neighborhood, provided that, as is surely possible, $y_1, \cdots, y_n$ are so chosen that they and their first $n - 1$ derivatives are continuous functions of (x, λ). Then (3) gives the general solution of the system (1) for all values of λ in a certain neighborhood of λ_0, and it is clear that for any special determination of the c_i's, either as constants or as continuous functions of λ, the function (3) is continuous in (x, λ). Hence

LEMMA I. *If throughout a certain range of values of λ the coefficients of L are continuous functions of (x, λ) and the coefficients of $U_1, \cdots, U_n$ are continuous functions of λ, and if for all values of λ in this range the system* (1) *has exactly k-fold compatibility; then if $u_0(x)$ denotes any particular solution of the system* (1) *when $\lambda = \lambda_0$, there exists a function $u(x, \lambda)$ continuous in (x, λ) which, throughout a certain neighborhood* of λ_0, satisfies* (1), *and is such that its limit for $\lambda = \lambda_0$ is $u_0(x)$, this limit being approached uniformly in ab.*

We turn next to

LEMMA II. *If v is any solution of the system*

$$(4) \qquad M(v) = 0, \qquad V_i(v) = 0 \quad (i = 1, 2, \cdots, n)$$

adjoint† *to* (1), *and u_0 is any solution of the system*

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* This will be a one-sided neighborhood if λ_0 is an extremity of the range in question.

† For a definition of the adjoint system cf. for instance the paper already cited, where a more detailed statement of Green's theorem will also be found.

$$(5) \qquad L(u) = gu, \qquad U_i(u) = 0 \quad (i = 1, 2, \cdots, n),$$

then

$$(6) \qquad \int_a^b g u_0 v \, dx = 0.$$

The proof consists in applying Green's theorem

$$\int_a^b [vL(u) - uM(v)]dx = \sum_{i=1}^{2n} U_i(u)V_{2n+1-i}(v)$$

to the two functions u_0 and v, when it reduces at once to (6).

LEMMA III. *If the system* (1) *has k-fold compatibility* ($k \geqq 1$), *and* ϵ *is an arbitrarily given positive constant, a continuous, real function* $g(x)$ *exists such that* $0 \leqq g(x) < \epsilon$, *and that the system* (5) *has less than k-fold compatibility.*

To prove this, let u be a non-identically vanishing solution of (1) and v a similar solution of (4), which surely exists since (1) and (4) always have the same order of compatibility.* Since, by a fundamental (though seldom explicitly stated) theorem concerning homogeneous linear differential equations, neither u nor v has more than a finite number of zeros in ab, we can select a point p at which the product uv does not vanish. Either the real or the pure imaginary part of uv does not vanish at p; and without loss of generality we may assume that the former is the case as otherwise we might have multiplied v by a pure imaginary constant before beginning. Since uv, and therefore its real part, is a continuous function of x, we can surround p by an interval $a'b'$ so short that the real part of uv does not vanish there. Now define φ as a real continuous function of x which vanishes everywhere outside of $a'b'$ and is positive but less than ϵ everywhere within. We see then that

$$(7) \qquad \int_a^b \varphi u v \, dx \neq 0.$$

We now define the function g, whose existence is asserted in our lemma, by the equation

$$g = \lambda\varphi,$$

where λ is an, as yet undetermined, positive constant less than 1; and we see from (6) that

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* Loc. cit., Theorem I.

$$(8) \qquad \int_a^b \varphi u_g v\, dx = 0,$$

where u_g is any solution of (5).

Now assume Lemma III to be false. Then for all positive values of λ less than 1 the system (5) would have at least k-fold compatibility, while by Theorem I it cannot have more than k-fold compatibility for sufficiently small values of λ. Let us then restrict λ to values so small that (5) has always exactly k-fold compatibility. Then, by Lemma I, we can take for u_g a continuous function of (x, λ) which approaches $u(x)$ uniformly as λ approaches zero through positive values. Consequently

$$\lim_{\lambda=+0} \int_a^b \varphi u_g v\, dx = \int_a^b \varphi u v\, dx.$$

This, however, is in contradiction with formulas (7) and (8). Thus our lemma is proved.

We have indeed proved more than is stated in the lemma, for we have shown that g may be taken as identically zero except in the interval $a'b'$, which interval could be taken as short as we please and in any position we please provided it avoids a finite number of points. If now the order of compatibility of (5) is not zero, we can start afresh with this system, in place of (1), and, applying Lemma III to it, form a new system

$$L(u) = gu + g_1 u, \qquad U_i(u) = 0 \quad (i = 1, 2, \cdots, n)$$

which has a still lower order of compatibility and where $0 \leqq g_1 < \epsilon$. Moreover g_1 can be made to vanish everywhere except in an interval $a''b''$ as short as we please and not overlapping the interval $a'b'$. Hence the function $g + g_1$ satisfies the same inequality as g and g_1. Proceeding in this way step by step, we finally come to a system which is incompatible. Since all the intervals $a'b'$, $a''b''$, etc., which we use may be taken, if we wish, within an arbitrarily chosen subinterval of ab, we may state our final result as follows:

THEOREM II. *If ϵ is an arbitrarily given positive constant, a continuous, real function $g(x)$ exists such that $0 \leqq g(x) < \epsilon$ and such that the system (5) is incompatible. This function g may be taken to be identically zero except in an arbitrarily chosen subinterval of ab.*

We have proved this theorem, it is true, only when $k > 0$. If $k = 0$ it is, however, merely an obvious consequence of Theorem I.

We come now at last to our most important result, though one which is, at bottom, less far reaching than Theorem II, namely

THEOREM III. *If* ϵ *is an arbitrarily given positive constant, a continuous, real function* $g(x)$ *exists such that* $0 < g(x) < \epsilon$ *and such that the system* (5) *is incompatible.*

The proof consists simply in noticing that if we add to the function $g(x)$ determined in Theorem II a sufficiently small function everywhere positive (not zero), the system (5) will, by Theorem I, remain incompatible.*

This theorem is useful in making connection, by the method originally given in special cases by Hilbert, between the system (1) and an integral equation of the second kind.

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THE SMALLEST CHARACTERISTIC NUMBERS IN A CERTAIN EXCEPTIONAL CASE.

BY PROFESSOR MAXIME BÔCHER

(Read before the American Mathematical Society, September 8, 1914.)

THE characteristic numbers of the system

$$\frac{d}{dx}(ku') + (\lambda g - l)u = 0, \quad (k > 0,\ l \geqq 0), \tag{1}$$

$$\alpha u'(a) - \alpha' u(a) = 0, \quad (\alpha\alpha' \geqq 0,\ |\alpha| + |\alpha'| > 0), \tag{2}$$

$$\beta u'(b) + \beta' u(b) = 0, \quad (\beta\beta' \geqq 0,\ |\beta| + |\beta'| > 0) \tag{3}$$

are those values of λ for which (1) has a solution not identically zero which satisfies (2) and (3). We assume that k, g, l are continuous real functions of x in the interval $a \leqq x \leqq b$, and that α, α', β, β' are real constants. It is with the real characteristic values we shall be concerned; indeed it is easily proved that there are no others.

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* A similar method enables us to deduce from Theorem II a great variety of other results, for instance:

If ϵ *is an arbitrarily given positive constant, and* $x_1, \cdots, x_p$ *are arbitrarily given points in* ab, *there exists a continuous, real function* $g(x)$ *which vanishes and changes sign at each of the points* x_i *but vanishes nowhere else in* ab, *which satisfies the condition* $|g(x)| < \epsilon$, *and for which* (5) *is incompatible.*

If $g \geqq 0$, but not identically zero, we have obviously a special case of the system considered by Sturm in his first memoir of 1836, and indeed one which Sturm, and later Liouville, considered explicitly. If g changes sign, we have a case which has been supposed to be essentially different from Sturm's, and has been several times treated on this basis by more or less independent methods. I believe I was the first to point out* that if we divide (1) by $|\lambda|$ the system becomes merely a special case of the one originally treated by Sturm, whose results tell us that the characteristic numbers consist of two infinite systems, λ_0^+, λ_1^+, $\cdots$ and λ_0^-, λ_1^-, $\cdots$, the first set positive, the second negative (we suppose these sets to be both arranged according to increasing absolute values), such that the characteristic function corresponding to any one of them has in the interval $a < x < b$ a number of zeros exactly equal to the subscript of the λ in question.

There is, however, one exceptional case in which the result just stated becomes accurate only after a slight modification, namely that in which

$$l \equiv 0, \qquad \alpha' = \beta' = 0.$$

In this case, and in this case only, $\lambda = 0$ is a characteristic number, a corresponding characteristic function being $u = 1$. In this case also Sturm's results assert the existence of the characteristic numbers λ_i^+ and λ_i^-, when $i \geqq 1$, and the fact that the corresponding characteristic functions vanish just i times. It is not, however, immediately obvious whether the numbers λ_0^+ and λ_0^- still exist. As a matter of fact at least one, and sometimes both of them, will have disappeared, or, if we prefer, will reduce to the value $\lambda = 0$. The facts in this special case were established by Picone† by a different method. I wish here to show how very readily they follow by means of Sturm's results.

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* In a lecture delivered at the 5th International Congress of Mathematicians at Cambridge, England, 1912. See Proceedings, vol. 1, pp. 173, 176. The literature there cited should have included a reference to Hilb, *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 16 (1907), p. 279, where a sketch for a treatment of a special case in which g changes sign is given, viz. $l \equiv 0$, $\alpha = \beta = 0$.

† *Annali della R. Scuola Normale di Pisa*, vol. 11 (1909), p. 39.

Let $v(x, \lambda)$ be the solution of (1) for which $v(a, \lambda) = 1$, $v'(a, \lambda) = 0$. According to Sturm's second theorem of comparison,* as λ increases from the value 0, the function

$$\frac{k(b)v'(b, \lambda)}{\lambda v(b, \lambda)} \tag{4}$$

decreases until it becomes negatively infinite. This occurs when λ reaches the value, μ_1, for which $v(b, \mu_1) = 0$. The characteristic number λ_0^+, if it exists, must evidently lie in the interval $0 < \lambda < \mu_1$, since a characteristic function corresponding to a characteristic number greater than μ_1 would necessarily have at least one zero in ab.

Now (4) approaches a definite finite limit as λ approaches zero, namely

$$D = k(b)\left[\frac{\partial}{\partial\lambda} v'(b, \lambda)\right]_{\lambda=0}. \tag{5}$$

Consequently, if $D > 0$, the quantity (4) in its decrease starts from positive values and passes, for some value of λ less than μ_1, through the value zero. This is the characteristic value λ_0^+, since for it $v'(b, \lambda) = 0$. On the other hand, if $D \leqq 0$, (4) does not pass through the value zero, and λ_0^+ does not exist.

In just the same way, by considering negative values of λ and dividing (1) by $|\lambda|$, we infer from Sturm's second theorem of comparison that, if $D < 0$, λ_0^- exists, if $D \geqq 0$, it does not.

It remains merely to evaluate D in terms of given quantities. From (1) we have

$$\frac{\partial}{\partial x}[k(x)v'(x, \lambda)] = -\lambda g(x)v(x, \lambda).$$

Integrating, and remembering that $v'(a, \lambda) = 0$, we get

$$k(b)v'(b, \lambda) = -\lambda \int_a^b g(x)v(x, \lambda)dx,$$

or, on differentiating with regard to λ,

$$k(b)\frac{\partial}{\partial\lambda} v'(b, \lambda) = -\int_a^b g(x)v(x, \lambda)dx - \lambda \int_a^b g(x)\frac{\partial}{\partial\lambda} v(x, \lambda)dx.$$

If here we let $\lambda = 0$, we get the desired formula

$$D = -\int_a^b g(x)dx. \tag{6}$$

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* See, for instance, *Transactions*, vol. 1 (1900), p. 420.

All the steps we have here taken are justified since $v(x, \lambda)$ and $v'(x, \lambda)$ are both continuous in (x, λ) and analytic in λ.[*]

We have thus established Picone's result:

The differential system

$$\frac{d}{dx}(ku') + \lambda g u = 0, \qquad (k > 0)$$

$$u'(a) = 0, \qquad u'(b) = 0$$

in which g changes sign in ab has, if $\int_a^b g\,dx = 0$, no characteristic number other than zero for which the characteristic function does not vanish, otherwise it has just one such characteristic number, namely a positive one if $\int_a^b g\,dx < 0$, a negative one if $\int_a^b g\,dx > 0$.

I note in closing that the case $\int_a^b g\,dx = 0$ is of interest as giving one of the simplest examples of a characteristic number ($\lambda = 0$) whose order of multiplicity when regarded as a root of the characteristic equation (2 in this case) is not equal to its index (1 in this case), i. e., the number of linearly independent characteristic functions corresponding to it.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.,
July 15, 1914.

ON APPROXIMATION BY TRIGONOMETRIC SUMS.

BY PROFESSOR T. H. GRONWALL.

(Read before the American Mathematical Society, December 31, 1913.)

IN his paper "On approximation by trigonometric sums and polynomials"[†] Dr. Jackson has shown that, $f(x)$ being a function of period 2π and satisfying the Lipschitz condition

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|$$

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[*] For the fundamental properties of this important class of functions see my paper "On semi-analytic functions of two variables," *Annals of Mathematics*, 2d ser., vol. 12 (1910), p. 18. I was not aware when I published this article that some of these properties had been already given by Dini, *Annali di Matematica*, ser. 3, vol. 12 (1906), p. 179.

[†] *Transactions*, vol. 13 (1912), pp 491–515.

for all values of x_2 and x_1, then there exists, for every integer n, a trigonometric sum of order not exceeding n

$$T_n(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx$$
$$+ b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx,$$

approximating $f(x)$ in such a way that for all values of x

$$| f(x) - T_n(x) | < \frac{4J'_m}{J_m} \cdot \frac{\lambda}{n},$$

where the integer m is determined by the condition $2m - 2 \leqq n < 2m$, and

$$(1) \qquad \begin{aligned} J_m &= \frac{1}{m^3} \int_0^{\pi/2} \left(\frac{\sin mu}{\sin u} \right)^4 du, \\ J'_m &= \frac{1}{m^2} \int_0^{\pi/2} u \left(\frac{\sin mu}{\sin u} \right)^4 du. \end{aligned}$$

By asymptotic considerations, Dr. Jackson shows that for $m \geqq 4$, $n \geqq 6$,

$$(2) \qquad \frac{4J'_m}{J_m} \leqq 2.90\cdots$$

It is the purpose of the present note to show that the quotient in (2) decreases as m increases,

$$(3) \qquad \frac{J'_{m+1}}{J_{m+1}} - \frac{J'_m}{J_m} < 0 \qquad (m = 1, 2, 3, \cdots).$$

Since it may be shown that*

$$(4) \qquad J_m = \frac{\pi}{3}\left(1 + \frac{1}{2m^2} \right),$$

the inequality (3) is equivalent to

$$(5) \qquad \begin{aligned} \Delta_m = (2m^2 + 1) \int_0^{\pi/2} u \, \frac{\sin^4 (m+1)u - \sin^4 mu}{\sin^4 u} \, du \\ - (4m + 3) \int_0^{\pi/2} u \left(\frac{\sin mu}{\sin u} \right)^4 du < 0. \end{aligned}$$

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* T. H. Gronwall, "On the degree of convergence of Laplace's series," *Transactions*, vol. 15 (1914), pp. 1–30.

We shall begin by replacing u by an approximate expression in terms of $\sin u$. We have

$$\frac{d}{du}\frac{u-\sin u}{\sin^3 u} = \frac{\sin u - 3u\cos u + 2\sin u\cos u}{\sin^4 u}$$

and

$$\frac{d}{du}(\sin u - 3u\cos u + 2\sin u\cos u) = 3\sin u(u-\sin u)$$
$$+ (1-\cos u)^2 > 0 \text{ for } 0 < u < \pi/2;$$

therefore $\sin u - 3u\cos u + 2\sin u\cos u > 0$ for $0 < u < \pi/2$, and consequently $\frac{u-\sin u}{\sin^3 u}$ increases monotonely with u for $0 < u < \pi/2$, so that

$$u = \sin u + \eta(u)\sin^3 u,$$

$$\frac{1}{6} = \eta(0) < \eta(u) < \eta\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 1 \text{ for } 0 < u < \frac{\pi}{2}. \tag{6}$$

The expression (5) for Δ_m now gives

$$\begin{aligned}\Delta_m = {} & (2m^2+1)\int_0^{\pi/2}\frac{\sin^4(m+1)u - \sin^4 mu}{\sin^3 u}\,du \\ & - (4m+3)\int_0^{\pi/2}\frac{\sin^4 mu}{\sin^3 u}\,du \\ & + (2m^2+1)\int_0^{\pi/2}\eta(u)\,\frac{\sin^4(m+1)u - \sin^4 mu}{\sin u}\,du \\ & - (4m+3)\int_0^{\pi/2}\eta(u)\,\frac{\sin^4 mu}{\sin u}\,du \\ < {} & (2m^2+1)\int_0^{\pi/2}\frac{\sin^4(m+1)u - \sin^4 mu}{\sin^3 u}\,du \\ & - (4m+3)\int_0^{\pi/2}\frac{\sin^4 mu}{\sin^3 u}\,du \\ & + (2m^2+1)\left(\frac{\pi}{2}-1\right)\int_0^{\pi/2}\frac{\sin^4(m+1)u - \sin^4 mu}{\sin u}\,du \\ & - \frac{4m+3}{6}\int_0^{\pi/2}\frac{\sin^4 mu}{\sin u}\,du.\end{aligned} \tag{7}$$

Proceeding to evaluate our various integrals, we find, integrating by parts,

$$\begin{aligned}\int_0^{\pi/2}\frac{\sin^4 mu}{\sin^3 u}\,du &= \int_0^{\pi/2}\frac{\sin^4 mu}{\sin u}\,d(-\cot u)\\ &= \int_0^{\pi/2}\cot u\,\frac{d}{du}\left(\frac{\sin^4 mu}{\sin u}\right)du\\ &= \int_0^{\pi/2}\frac{4m\sin^3 mu\cos mu\cos u}{\sin^2 u}\,du\\ &\qquad -\int_0^{\pi/2}\frac{\sin^4 mu}{\sin^3 u}\,du+\int_0^{\pi/2}\frac{\sin^4 mu}{\sin u}\,du,\end{aligned}$$

or

$$\begin{aligned}\int_0^{\pi/2}\frac{\sin^4 mu}{\sin^3 u}\,du &= 2m\int_0^{\pi/2}\sin^3 mu\cos mu\,d\left(-\frac{1}{\sin u}\right)\\ &\qquad +\frac{1}{2}\int_0^{\pi/2}\frac{\sin^4 mu}{\sin u}\,du\\ &= 2m\int_0^{\pi/2}\frac{1}{\sin u}\frac{d}{du}(\sin^3 mu\cos mu)du\\ &\qquad +\frac{1}{2}\int_0^{\pi/2}\frac{\sin^4 mu}{\sin u}\,du\\ &= 2m^2\int_0^{\pi/2}\frac{\sin^2 2mu-\sin^2 mu}{\sin u}\,du\\ &\qquad +\frac{1}{2}\int_0^{\pi/2}\frac{\sin^2 mu-\frac{1}{4}\sin^2 2mu}{\sin u}\,du.\end{aligned}$$

Now the identity

$$\frac{\sin^2 nu}{\sin u}=\sum_{\lambda=0}^{n-1}\sin(2\lambda+1)u$$

gives

$$\int_0^{\pi/2}\frac{\sin^2 nu}{\sin u}\,du=\sum_{\lambda=0}^{n-1}\frac{1}{2\lambda+1},$$

and consequently

$$\int_0^{\pi/2}\frac{\sin^4 mu}{\sin^3 u}\,du = 2m^2\sum_{\lambda=m}^{2m-1}\frac{1}{2\lambda+1}+\frac{1}{2}\sum_{\lambda=0}^{m-1}\frac{1}{2\lambda+1}$$

$$(8)\qquad\qquad -\frac{1}{8}\sum_{\lambda=0}^{2m-1}\frac{1}{2\lambda+1},$$

$$\int_0^{\pi/2}\frac{\sin^4 mu}{\sin u}\,du = \sum_{\lambda=0}^{m-1}\frac{1}{2\lambda+1}-\frac{1}{4}\sum_{\lambda=0}^{2m-1}\frac{1}{2\lambda+1}.$$

From these equations and the last part of (7) we obtain

$$\Delta_m < -\,2(m^2-2m-1)\sum_{\lambda=m}^{2m-1}\frac{1}{2\lambda+1}$$

$$-\frac{2}{3}(4m+3)\left(\sum_{\lambda=0}^{m-1}\frac{1}{2\lambda+1}-\frac{1}{4}\sum_{\lambda=0}^{2m-1}\frac{1}{2\lambda+1}\right)$$

$$+\,(2m^2+1)\left(\frac{\frac{\pi}{2}-1}{2m+1}+\frac{1-\frac{1}{4}\left(\frac{\pi}{2}-1\right)}{4m+1}-\frac{1}{4}\cdot\frac{1}{4m+3}\right).$$

For $m > 2$, the first term in the expression on the right is negative, and since the difference of the two sums in the second term obviously increases with m, we have for $m > 2$

$$\sum_{\lambda=0}^{m-1}\frac{1}{2\lambda+1}-\frac{1}{4}\sum_{\lambda=0}^{2m-1}\frac{1}{2\lambda+1} > 1+\tfrac{1}{3}-\tfrac{1}{4}(1+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7})=\tfrac{32}{35};$$

furthermore

$$\frac{\pi-2}{4m+2}+\frac{1-\frac{1}{4}\left(\frac{\pi}{2}-1\right)}{4m+1}-\frac{1}{4}\cdot\frac{1}{4m+3}$$

$$< \frac{\pi-2+1-\frac{1}{4}\left(\frac{\pi}{2}-1\right)}{4m+1} < \frac{2}{4m+1},$$

so that finally, for $m > 2$,

$$\Delta_m < -\,\tfrac{2}{3}\cdot\tfrac{32}{35}(4m+3)+\frac{4m^2+2}{4m+1} < 0,$$

which proves our theorem for $m > 2$. For $m = 1$ and $m = 2$ it is readily verified from the numerical values of J_m and J'_m given by Jackson.

From (1), (6) and (8) it follows that

$$J'_m = \frac{1}{m^2}\int_0^{\pi/2} \frac{\sin^4 mu}{\sin^3 u}\,du + \eta' \cdot \frac{1}{m^2}\int_0^{\pi/2} \frac{\sin^4 mu}{\sin u}\,du,$$

$$\tfrac{1}{6} < \eta' < \left(\frac{\pi}{2} - 1\right),$$

and consequently

$$J'_m = 2\sum_{\lambda=m}^{2m-1} \frac{1}{2\lambda+1} + O\left(\frac{\log m}{m^2}\right),$$

$$\lim_{m=\infty} J'_m = \lim_{m=\infty} \sum_{\lambda=1}^{2m-1} \frac{2}{2\lambda+1} = \int_0^1 \frac{dx}{1+x} = \log 2.$$

Using this result in connection with (4), it is seen that

$$\lim_{m=\infty} \frac{4J'_m}{J_m} = \frac{12 \log 2}{\pi} = 2.648 - ,$$

so that, using the numerical values of J, and J',, we may finally state the result

$$2.758 - = \frac{4J_4'}{J_4} > \frac{4J_5'}{J_5} > \frac{4J_6'}{J_6} > \cdots > 2.648 - .$$

Princeton University,
December 20, 1913.

NOTE ON THE ROOTS OF ALGEBRAIC EQUATIONS.

BY PROFESSOR R. D. CARMICHAEL AND DR. T. E. MASON.

(Read before the American Mathematical Society at Chicago, April 10, 1914.)

1. Landau* has established certain interesting inequalities concerning the least root of a class of algebraic equations, having been led to these results by considerations connected with his remarkable extension and generalization of Picard's famous theorem to the effect that an entire function which fails to assume two values is a constant. These special in-

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* *Annales de l'École Normale Supérieure* (3), vol. 24 (1907), pp. 179–201; *Vierteljahrsschrift der Naturf. Gesellschaft in Zurich*, vol. 51 (1906), pp. 252–318.

equalities have been generalized and made more precise by Allardice.[*] Finally, still more comprehensive results have been obtained by Fejér[†] by a method which leaves nothing to be desired in the way of simplicity and elegance. Fejér's result, which contains those of the other authors mentioned, may be stated as follows:

Let

$$(1) \qquad a_0 + a_1 z^{\nu_1} + a_2 z^{\nu_2} + \cdots + a_k z^{\nu_k} = 0$$

be an equation of $k + 1$ *terms in which*

$$a_0 \neq 0, \; a_1 \neq 0, \; \nu_1 < \nu_2 < \cdots < \nu_k.$$

Let ζ *be a root of this equation of least absolute value. Then we have the inequalities*

$$(2) \quad |\zeta| \leqq \left[\frac{\nu_2 \, \nu_3 \cdots \nu_k}{(\nu_2 - \nu_1)(\nu_3 - \nu_1) \cdots (\nu_k - \nu_1)}\right]^{1/\nu_1} \left|\frac{a_0}{a_1}\right|^{1/\nu_1},$$

$$(3) \quad |\zeta| \leqq \left[\frac{(\nu_1 + 1)(\nu_1 + 2) \cdots (\nu_1 + k - 1)}{1 \cdot 2 \cdots (k-1)}\right]^{1/\nu_1} \left|\frac{a_0}{a_1}\right|^{1/\nu_1}.$$

In case $\nu_1 = 1$, *we have*

$$(4) \qquad |\zeta| \leqq k \left|\frac{a_0}{a_1}\right|;$$

and this value is actually attained by the roots of the equation

$$a_0 \left(1 + \frac{a_1 z}{k a_0}\right)^k = 0.$$

For the case $\nu_1 = 1$ formula (2) was obtained by Allardice. For the case when $k = 2$ and $\nu_1 = 1$ formula (4) was established by Landau. Furthermore, for $k = 3$ and $\nu_1 = 1$ Landau proved that $|\zeta|$ is not greater than $5\frac{2}{3} a_0/a_1$, a result less precise than that of Fejér above.

The principal purpose of this note is to establish other results concerning the least roots of algebraic equations, especially of those of certain special forms. It will be seen that these results are in some cases more far-reaching than those of Fejér. An additional theorem of a related nature is also given in the final section.

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[*] BULLETIN, vol. 13 (1907), pp. 443–447.

[†] *Comptes rendus*, vol. 145 (1907), pp. 459–461.

It is convenient, first of all, to simplify equation (1) in the following way: Divide each member by a_0 and replace z by x, where

$$x = \left(\frac{a_1}{a_0}\right)^{1/\nu_1} z.$$

Then the equation takes the form

$$(5) \qquad 1 + x^{\nu_1} + \alpha_2 x^{\nu_2} + \alpha_3 x^{\nu_3} + \cdots + \alpha_k x^{\nu_k} = 0.$$

We shall confine our attention principally to the latter equation.

2. We shall consider first the case in which $\nu_1 = 1$ and $\nu_{i+1} = \nu_i + 1$ when $i > 1$, writing the equation in the form

$$(6) \qquad 1 + x + c_{s+1}x^{s+1} + c_{s+2}x^{s+2} + \cdots + c_m x^m = 0 \quad (s \geqq 1).$$

Let $x_1, x_2, \cdots, x_m$ denote the roots of this equation. From the equation whose roots are the reciprocals of those of (6) we see at once that

$$\Sigma \frac{1}{x_1} = -1,\ \Sigma \frac{1}{x_1 x_2} = 0,\ \cdots,\ \Sigma \frac{1}{x_1 x_2 \cdots x_s} = 0,$$

where the summation in each case is for all terms of the type written. From these equations and the customary formulas for the sum of the roots of an equation in terms of the elementary symmetric functions of these roots* it follows readily that

$$\Sigma \frac{1}{x_1^s} = (-1)^s.$$

Therefore

$$\frac{1}{|x_1|^s} + \frac{1}{|x_2|^s} + \cdots + \frac{1}{|x_m|^s} \geqq 1.$$

Now if we suppose that x_1 is a root of (6) of least absolute value, we have

$$\frac{m}{|x_1|^s} \geqq 1;$$

whence

$$|x_1| \leqq \sqrt[s]{m}.$$

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* See Bôcher's Higher Algebra, p. 244.

This result may be stated in the form of the following theorem:

I. *The equation*

$$1 + x + c_{s+1}x^{s+1} + c_{s+2}x^{s+2} + \cdots + c_m x^m = 0 \quad (s \geqq 1),$$

has a root which is not greater in absolute value than $\sqrt[s]{m}$, *whatever values* $c_{s+1}, c_{s+2}, \cdots, c_m$ *may have.*

COROLLARY. *The equation*

$$(7) \qquad 1 + x + c_{s+1}x^{s+1} + c_{s+2}x^{s+2} + \cdots + c_{2^s}x^{2^s} = 0 \quad (s \geqq 1),$$

has a root which is not greater than 2 in absolute value.

The result of this corollary for the special case when $c_{s+2} = c_{s+3} = \cdots = c_{2^s} = 0$ is given by Landau (loc. cit.). Here the equation reduces to a trinomial.

If we apply to equation (7) the Allardice-Fejér formula (2) for $\nu_1 = 1$ we conclude that this equation has a root not greater in absolute value than $2^s/s$. For s approaching infinity this value $2^s/s$ itself approaches infinity at a rapid rate. Hence, the circle in which, according to the Allardice-Fejér result, equation (7) certainly has a root increases in size with increasing s. But from the theorem of our corollary it is seen that (7) always has a root in the circle of radius 2 about the point zero. Thus for the special case of equation (7) (and likewise of (6), as one may readily show) our results reach further than those of Allardice and Fejér.

We shall now show inversely that there is in general but one root of equation (6) which is bounded by the circle about zero of radius $\sqrt[s]{m}$ when $c_{s+1}, \cdots, c_m$ are arbitrary. More precisely, we shall prove the following theorem:

II. *Let s and m be any two positive integers having a prime number p between them: $s < p < m$. Let M be any positive constant. Then there exist equations of the form*

$$1 + x + c_{s+1}x^{s+1} + \cdots + c_m x^m = 0$$

having every root but one greater than M in absolute value.

To prove this theorem it is sufficient to construct equations having the specified property. Let kp ($k \geqq 1$) be the greatest multiple of p which is less than m. We shall further specialize the equation to be constructed so that it shall have the form

$$(8) \qquad 1 + x + c_p x^p + c_{p+1}x^{p+1} + \cdots + c_{kp+1}x^{kp+1} = 0.$$

Now let ω be any primitive pth root of unity; then the set of pth roots of unity is $1, \omega, \omega^2, \cdots, \omega^{p-1}$. Let $x_1, x_2, \cdots, x_{kp+1}$ denote the $kp+1$ roots of equation (8). We shall build up equation (8) by properly choosing the values of its roots. Thus, we put

$$x_{kp+1} = -1,$$
$$x_{\mu k+\nu} = \omega^{-\mu}\tau, \quad (\mu = 0, 1, \cdots, p-1;\ \nu = 1, 2, \cdots, k),$$

where τ is any number whatever which is different from zero.

With these values of $x_1, x_2, \cdots, x_{kp+1}$ we have

$$\Sigma\frac{1}{x_1{}^t} = (-1)^t \quad (t = 1, 2, \cdots, p-1),$$

as one sees readily by means of the well-known relations

$$1^t + \omega^t + \omega^{2t} + \cdots + \omega^{(p-1)t} = 0 \quad (t = 1, 2, \cdots, p-1).$$

From the usual formulas (Bôcher, loc. cit.) for the sum of the roots of an equation in terms of the elementary symmetric functions of these roots it is now easy to see that

$$\Sigma\frac{1}{x_1} = -1, \quad \Sigma\frac{1}{x_1x_2} = 0, \quad \Sigma\frac{1}{x_1x_2x_3} = 0, \cdots, \quad \Sigma\frac{1}{x_1x_2\cdots x_{p-1}} = 0.$$

Hence, the equation of which the roots are $x_1, x_2, \cdots, x_{kp+1}$ has the form (8). If $|\tau| > M$, then all but one of the roots of the equation so formed are greater than M in absolute value. Hence we conclude to the theorem as stated above.

By means of II and the corollary to I we shall now prove the following theorem:

III. *Every equation of the form*

$$1 + x + c_{s+1}x^{s+1} + c_{s+2}x^{s+2} + \cdots + c_{2^s}x^{2^s} = 0 \quad (s \geqq 1),$$

has at least one root not greater than 2 in absolute value, while special equations of this form may have all roots but one greater in absolute value than any preassigned M.

In order to complete the demonstration of this result it is sufficient to observe that obviously only one root is bounded when $s = 1$ and that a prime number p lies between s and 2^s when $s \geqq 2$. For $s = 2$ we have $p = 3$ and for $s = 3$ we have $p = 5$ or 7. For $s > 3$ we may prove the existence of p by means of Tschebyschef's theorem*: If an integer s is greater than 3 then there is at least one prime number between s and $2s - 2$.

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* Bachmann, Niedere Zahlentheorie I, p. 66.

3. Next, let us consider the equation

$$1 + x^r + c_{sr+1}x^{sr+1} + \cdots + c_m x^m = 0 \qquad (s \geqq 1). \tag{9}$$

Let $x_1, x_2, \cdots, x_m$ be the roots of this equation and denote by A_k and S_k respectively, the sums

$$A_k = \Sigma \frac{1}{x_1 x_2 \cdots x_k}, \quad S_k = \Sigma \frac{1}{x_1{}^k}.$$

Then we have*

$$\begin{aligned} S_m - A_1 S_{m-1} + A_2 S_{m-2} - \cdots \\ + (-1)^{m-1} A_{m-1} S_1 + (-1)^m m A_m = 0. \end{aligned} \tag{10}$$

Now from equation (9) we see readily that

$$A_1 = A_2 = \cdots = A_{r-1} = 0, \;\; A_r = (-1)^r, \;\; A_{r+1} = \cdots = A_{sr} = 0.$$

Hence it follows from (10) that

$$S_r = -r;$$

and then that

$$S_{2r} = r, \text{ if } s \geqq 2.$$

Continuing thus we have finally

$$S_{sr} = (-1)^s r;$$

that is,

$$\frac{1}{x_1{}^{sr}} + \frac{1}{x_2{}^{sr}} + \cdots + \frac{1}{x_m{}^{sr}} = (-1)^s r.$$

If x_1 is a root of (9) of least absolute value, then from the last equation it follows readily that

$$\frac{m}{|x_1|^{sr}} \geqq r \quad \text{or} \quad |x_1| \leqq \sqrt[sr]{\frac{m}{r}}.$$

Thus we have the following generalization of theorem I:

IV. *Every equation of the form* (9) *has at least one root which is less in absolute value than* $\sqrt[sr]{m/r}$.

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* See Bôcher's Higher Algebra, p. 244.

We may also generalize theorem II and so obtain the following result:

V. *Let r, s and m be any three positive integers such that r is prime while a prime number p lies between sr and $m - r + 1$; $sr < p < m - r + 1$. Let M be any positive constant. Then there exist equations of the form* (9) *which have all but r of their roots greater than M in absolute value.*

The detailed proof of this theorem will not be given. It is sufficient to construct an equation of the form

$$1 + x^r + c_p x^p + c_{p+1}x^{p+1} + \cdots + c_{kp+r}x^{kp+r} = 0 \quad (kp + r \leqq m),$$

by means of its roots $x_1, x_2, \cdots, x_{kp+r}$ defined as follows:

$$x_{\mu k+\nu} = \omega^{-\mu}\tau, \ (\mu = 0, 1, 2, \cdots, p-1; \ \nu = 1, 2, \cdots, k);$$

$$x_{kp+t} = \eta\epsilon^{-t} \quad (t = 1, 2, \cdots, r),$$

ω being a primitive pth root of unity, ϵ a primitive rth root of unity and η any rth root of -1.

4. Related to the foregoing theorems is the following rather obvious one:

VI. *In the equation*

$$(11) \qquad 1 + a_1x + \cdots + a_kx^k + \cdots + a_mx^m = 0,$$

of degree m, let a_k be any given non-zero coefficient and let all the other coefficients be chosen in any manner whatever. The equation has at least one root which is not greater in absolute value than

$$(12) \qquad \sqrt[k]{\frac{m!}{k!(m-k)!}} \cdot \sqrt[k]{\frac{1}{|a_k|}}.$$

The equation

$$\left(1 + x\sqrt[k]{\frac{k!(m-k)!a_k}{m!}}\right)^m = 1 + a_1x + \cdots + a_kx^k + \cdots + a_mx^m = 0$$

has all its roots equal in absolute value to the quantity (12).

Let $x_1, x_2, \cdots, x_m$ denote the roots of the given equation and let x_1 be a root of least absolute value. We have readily

$$(13) \qquad \Sigma\frac{1}{x_1x_2\cdots x_k} = (-1)^k a_k;$$

whence

$$\frac{1}{|x_1|^k}\frac{m!}{k!(m-k)!} \geqq |a_k|,$$

since the number of terms in the first member of (13) is obviously equal to the number of combinations of m things taken k at a time. Hence,

$$|x_1| \leqq \sqrt[k]{\frac{m!}{k!(m-k)!}}\sqrt[k]{\frac{1}{|a_k|}}.$$

This proves the first part of the theorem. The second part is obviously true.

5. We shall now prove a theorem of a nature somewhat different from that of those in the preceding sections; namely, the following:

VII. *All the roots of the equation*

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0 \tag{14}$$

are in absolute value less than or equal to

$$\sqrt{1 + |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2}.$$

Let ζ be a root of least absolute value of the equation

$$1 + a_1z + a_2z^2 + \cdots + a_nz^n = 0.$$

Write

$$\frac{1}{1 + a_1z + a_2z^2 + \cdots + a_nz^n} = 1 + c_1z + c_2z^2 + \cdots. \tag{15}$$

Now, the circle of convergence of the power series in the second member of (15) passes through the point ζ. Its radius is therefore equal to $|\zeta|$. Therefore, by a well-known property of power series, we have

$$\frac{1}{|\zeta|} = \limsup_{m=\infty} \sqrt[m]{c_m}.$$

Hence, if $\bar{x}$ is a root of (14) of greatest absolute value, it is clear that

$$|x| = \limsup_{m=\infty} \sqrt[m]{c_m}. \tag{16}$$

Now, the value of c_m is readily expressed in the form of a determinant, as follows: If we multiply equation (15) through

p17552

by the denominator of its first member and in the result equate coefficients of like powers of z, we have

$$\begin{aligned} c_1 &= -a_1, \\ a_1c_1 + c_2 &= -a_2, \\ a_2c_1 + a_1c_2 + c_3 &= -a_3, \\ a_3c_1 + a_2c_2 + a_1c_3 + c_4 &= -a_4, \\ \cdots\cdots&\cdots\cdots. \end{aligned}$$

Hence

$$c_m = -\begin{vmatrix} 1 & 0 & 0 & 0 \cdots 0 & a_1 \\ a_1 & 1 & 0 & 0 \cdots 0 & a_2 \\ a_2 & a_1 & 1 & 0 \cdots 0 & a_3 \\ a_3 & a_2 & a_1 & 1 \cdots 0 & a_4 \\ \cdot & \cdot & \cdot & \cdot \quad \cdot & \cdot \end{vmatrix}. \tag{17}$$

Now, if Δ_m is a determinant of order m,

$$\Delta_m = \begin{vmatrix} a_{11} & \cdots & a_{1m} \\ \cdot & \cdot & \cdot \\ a_{m1} & \cdots & a_{mm} \end{vmatrix},$$

then

$$|\Delta_m| \leqq \sqrt{\sigma_1\sigma_2 \cdots \sigma_m},$$

where

$$\sigma_i = \sum_{j=1}^{m} |a_{ij}|^2.$$

This fundamental theorem is due to Hadamard.*

From this theorem and equation (17) we see readily that

$$|c_m| < \{1 + |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2\}^{m/2}, \tag{18}$$

since each row in the determinant in (17) has the property that the sum of the squares of the absolute values of its elements is not greater than $1 + |a_1|^2 + \cdots + |a_n|^2$. From (18) and the result associated with (16) the theorem now follows immediately.

INDIANA UNIVERSITY,
February, 1914.

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* *Bull. des Sciences Math.* (Darboux), vol. 17 (1893), pp. 240–246.

REMARKS ON FUNCTIONAL EQUATIONS.

BY MR. A. R. SCHWEITZER.

(Read before the American Mathematical Society, September 8, 1913.)

1. The remarks of this article are in continuation of our paper in the Bulletin, volume 19 (1912), pages 66–70. We have mentioned that a class of functional equations arises from the solution of the equation

$$(1) \qquad \frac{\partial f(x, y)}{\partial x} + \mu(x, y)\frac{\partial f(x, y)}{\partial y} = 0,$$

where in particular we have assumed that $\mu(x, y) = \psi'(x)/\psi'(y)$. It is obvious that other classes of functional relations are obtained by considering the solution of the equation

$$\frac{dy}{dx} = \mu(x, y)$$

in connection with (1) when $\mu(x, y)$ is suitably specialized. The equation (1) may profitably be compared with another source of functional equations, that is, the equation*

$$(2) \qquad \frac{\partial \varphi(x, y)}{\partial x} + \frac{\partial \varphi(x, y)}{\partial y} = \lambda(x, y).$$

The following functional relation is derivable from (1) and admits an explicit solution:†

$$(1') \qquad \psi(x) - \psi(y) = \Omega^{-1}\{x\phi(y) - y\phi(x)\}.$$

This equation can be solved in another manner by noticing that $\phi(x)$ can be formally eliminated. The result is

$$(1'') \qquad \Omega\{x - y\} = \Omega(x)\psi_1^{-1}(y) - \Omega(y)\psi_1^{-1}(x),$$

where

$$\Omega(0) = 0, \quad \psi_1^{-1}(0) = 1,$$

$$\psi_1^{-1}(x) = \psi^{-1}\{x + \psi(1)\}.$$

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* Cf. V. Volterra, *Acc. Lincei*, vol. 19 (1910), pp. 169, 425; Bulletin, vol. 19, p. 171; G. C. Evans, Proc. Cong. Math., Cambridge, 1913, vol. 1, p. 387; J. Hadamard, *L'Enseignement Mathématique*, vol. 14 (1912), p. 18, note; Paul Lévy, Paris thesis.

† Cf. Bulletin, l. c., page 67.

We can now proceed as indicated by Abel, Œuvres, volume 1, page 10.

2. We consider the relations[*]

$$f(x, y) = \chi_f^{-1}\{\chi_f(x) - \chi_f(y)\}. \tag{3}$$

$$\phi(x, y) = \chi_f^{-1}\{\chi_f(x) + \chi_f(y)\}. \tag{4}$$

In (3) and (4) we write $\chi_f^{-1}(x)$ and $\chi_f^{-1}(y)$ for x and y respectively; then

$$f[\chi_f^{-1}(x), \chi_f^{-1}(y)] = \chi_f^{-1}\{x - y\}. \tag{3'}$$

$$\phi[\chi_f^{-1}(x), \chi_f^{-1}(y)] = \chi_f^{-1}\{x + y\}. \tag{4'}$$

Thus (3′) and (4′) represent respectively a subtraction formula and an addition formula.[†] It is thus seen that under differentiation, etc., symmetric and quasi-transitive[‡] functional properties give rise to addition and subtraction formulas. In the philosophy of mathematics this result is of considerable interest. In symmetry and transitiveness and formal properties closely allied to these, many mathematical disciplines are rooted, e.g., the foundations of geometry and the theory of quaternions. The fact that certain properties of the indicated category induce addition and subtraction formulas tends to corroborate the fundamental character of the latter in the theory of functions.

3. We have shown[§] purely formally that if

$$\phi\{f(y, z), z\} = y, \quad f\{\phi(y, z), y\} = z, \tag{5}$$

then ϕ can be eliminated; we obtained

$$f\{y, f(y, z)\} = z. \tag{5'}$$

In a similar manner f can be eliminated; for we have

$$\phi\{f[\phi(z, y), z], z\} = \phi(z, y),$$

that is,

$$\phi(y, z) = \phi(z, y). \tag{5''}$$

The relation (5″) may be used in the proof of theorem 1,

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[*] BULLETIN, l. c., p. 68.

[†] On this interpretation see Bourlet, *Ann. Sci. de l'Ecole Norm. Sup.*, vol. 14 (1897), pp. 140, 141. Independently of this reference, the preceding remark was kindly suggested to the author by Professor E. B. Van Vleck.

[‡] BULLETIN, vol. 18 (1912), p. 300.

[§] BULLETIN, vol. 19 (1912), p. 69.

Bulletin, l. c., page 68, as expeditiously as relation (5′) in theorem 2, l. c., page 69. With regard to the duality, as it were, between the two theorems it is desirable to replace the hypothesis of theorem 1

$$\phi\{x, \phi(y, z)\} = \phi\{y, \phi(x, z)\}$$

by the relation

$$\phi\{y, \phi(x, z)\} = \phi\{z, \phi(x, y)\}.$$

The correspondence between (5′) and (5″) is possibly of use elsewhere in analysis.

From the equations

$$f\{y, f(x, z)\} = f\{z, f(x, y)\}, \tag{6}$$

$$\phi\{f(y, z), z\} = y, \quad f\{\phi(y, z), y\} = z,$$

we have shown l. c., page 69, that the ϕ may be formally eliminated; the result is

$$f\{f(x, z), f(x, y)\} = f(y, z). \tag{7}$$

We proceed to eliminate f from the equations (6). Substituting $\phi(x, z)$ for x in the first of equations (6) we get

$$f(y, x) = f\{z, f[\phi(x, z), y]\}.$$

Hence,

$$f\{\phi(y, z), x\} = f\{z, f[\phi(x, z), \phi(y, z)]\}.$$

Therefore*

$$f\{\phi[y, \phi(x, z)], x\} = \phi(y, z).$$

Substituting in this equation $f(x, z)$ for x we find

$$f\{\phi(y, x), f(x, z)\} = \phi(y, z). \tag{8}$$

Now

$$\phi\{f[\phi(y, x), f(x, z)], f(x, z)\} = \phi(y, x).$$

Therefore by (8),

$$\phi\{\phi(y, z), f(x, z)\} = \phi(y, x). \tag{9}$$

In (9) we write $\phi(z, x)$ for x; then

$$\phi\{\phi(y, z), x\} = \phi\{y, \phi(z, x)\}$$

or

$$\phi\{x, \phi(y, z)\} = \phi\{y, \phi(x, z)\}. \tag{10}$$

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[*] From (6) we have $f\{f(y, z), f(x, z)\} = f(y, x)$ and consequently $x, y) = f\{\varphi(x, z), \varphi(y, z)\}$.

Hence Abel's functional equations* result by eliminating f from equations (6).

In a similar manner we may eliminate ϕ from the equations

$$\phi\{y, \phi(x, z)\} = \phi\{z, \phi(x, y)\}, \tag{11}$$

$$\phi\{f(y, z), z\} = y, \quad f\{\phi(y, z), y\} = z.$$

To do this, we easily derive from equations (11) the equation (9); thence using the relation

$$f\{\phi[f(z, y), \phi(x, y)], \quad f(z, y)\} = \phi(x, y)$$

we obtain equation (8). From (9) there results in connection with (11)

$$\phi\{f(x, y), z\} = \phi\{f(z, y), x\}. \tag{12}$$

And from (8) we derive

$$f\{z, f(y, x)\} = \phi\{f(z, y), x\}. \tag{13}$$

From (12) and (13) we get finally the desired relation

$$f\{x, f(y, z)\} = f\{z, f(y, x)\}. \tag{14}$$

The preceding formal eliminations enter into Grassmann's theory of synthetic and analytic associations which, as is well known, provides an abstract generalization of the processes of addition and subtraction.† In fact we may regard $\phi(x, y)$ and $f(x, y)$ as instances of Grassmann's associations $(x \cap y)$ and $(x \cup y)$ respectively; we have then in Grassmann's theory the theorems

I. $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$

imply

$(y \cap z) = (z \cap y)$ and $\{y \cup (y \cup z)\} = z.$

II. $\{y \cup (x \cup z)\} = \{z \cup (x \cup y)\}$

and $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$

imply

$\{y \cap (x \cap z)\} = \{z \cap (x \cap y)\}.$

III. $\{y \cap (x \cap z)\} = \{z \cap (x \cap y)\}$

and $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$

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[*] Œuvres, vol. 1, p. 61.

[†] Cf. Gesammelte Werke, vol. 1, part 1, pp. 33–45.

imply

$$\{y \cup (x \cup z)\} = \{z \cup (x \cup y)\}.$$

IV.
$$\{y \cup (x \cup z)\} = \{z \cup (x \cup y)\}$$
$$\text{and } \{y \cup (y \cup z)\} = z$$

imply

$$\{(x \cup y) \cup (x \cup z)\} = (z \cup y).$$

The study of the preceding relationship naturally would lead one to consider the monograph of Pietzker, Beiträge zur Funktionenlehre, Leipzig, 1899.*

4. The reference to Grassmann† in the preceding section suggests the following problem:

Given the quasi-transitive function $f(x, y)$ to find the quasi-transitive function $f_1(x, y)$ such that

$$f_1\{f(x, y), z\} = f\{f_1(x, z), f_1(y, z)\}. \tag{15}$$

We have

$$f(x, y) = \chi^{-1}\{\chi(x) - \chi(y)\}, \tag{16}$$
$$f_1(x, y) = \chi_1^{-1}\{\chi_1(x) - \chi_1(y)\}.$$

In (15) we put $z = z_0$, a constant; then if $\psi(x) = f_1(x, z_0)$ we have

$$\psi\{f(x, y)\} = f\{\psi(x), \psi(y)\}. \tag{15'}$$

Substituting in this relation for $f(x, y)$ from (16) we obtain

$$\chi\psi\chi^{-1}(x - y) = \chi\psi\chi^{-1}(x) - \chi\psi\chi^{-1}(y). \tag{17}$$

Hence

$$\psi(x) = \chi^{-1}\{c\chi(x)\}, \tag{18}$$

where c is an arbitrary constant. On the other hand, we put in relation (16)

$$\chi_2(x) = c'\chi_1(x);$$

then

$$f_1(x, z) = \chi_2^{-1}\{\chi_2(x) - c'\chi_1(z)\} \tag{16'}$$

or for $z = z_0$

$$\psi(x) = \chi_2^{-1}\{\chi_2(x) - c''\}, \tag{19}$$

where c'' is an arbitrary constant. As is well known, functional relations of the types (18) and (19) have been treated by Schroeder and Abel respectively.* From (18) and (19) we have

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* Cf. Moritz, *Amer. Journ. of Mathematics*, vol. 24, No. 3; A. F. Carpenter, same journal, vol. 35, No. 1.

† L. c., pp. 41–42.

$$\chi^{-1}\{c\chi(x)\} = \chi_2^{-1}\{\chi_2(x) - c''\}$$

or

$$\chi_2\chi^{-1}(cx) = \chi_2\chi^{-1}(x) - c''$$

or

$$\chi_2\chi^{-1}(cx) = \chi_2\chi^{-1}(c) + \chi_2\chi^{-1}(x) - \chi_2\chi^{-1}(1). \tag{20}$$

The solution of (20) is

$$\chi_2\chi^{-1}(x) = \log_a c_1 x, \tag{21}$$

where c_1 is a constant and the base a is arbitrary. From (21) we have

$$\chi_2(x) = \log_a [c_1\chi(x)].$$

Hence

$$f_1(x, y) = \chi^{-1}\{c_1' \cdot \chi(x)/\chi(y)\} \tag{22}$$

which is the solution required.

Proceeding in a precisely analogous manner we can determine the symmetric function $\phi_1(x, y)$ such that

$$\phi_1\{\phi(x, y), z\} = \phi\{\phi_1(x, z), \phi_1(y, z)\},$$

where ϕ is a given symmetric function; we find

$$\phi_1(x, y) = \chi^{-1}\{c_2\chi(x) \cdot \chi(y)\}, \tag{23}$$

where c_2 is a constant and

$$\phi(x, y) = \chi^{-1}\{\chi(x) + \chi(y)\}.$$

A formal solution of a problem closely related to the above has been given by Pietzker, l. c., pages 9–10.

5. If we assume properties (6), (15) and the relation

$$f_1\{\phi_1(y, z), y\} = z, \tag{24}$$

then there exists a function $\chi(x)$ such that

$$\begin{aligned} \phi(x, y) &= \chi^{-1}\{\chi(x) + \chi(y)\}, \\ f(x, y) &= \chi^{-1}\{\chi(x) - \chi(y)\}, \\ f_1(x, y) &= \chi^{-1}\{\chi(x)/\chi(y)\}, \\ \phi_1(x, y) &= \chi^{-1}\{\chi(x) \cdot \chi(y)\}. \end{aligned} \tag{25}$$

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* Cf. M.-J. van Uven, *Kon. Akad. van Wetenschappen te Amsterdam*, vol. 12 (1909–1910), pp. 208, 427; A. Hurwitz, *Math. Annalen*, vol. 70 (1910–1911), p. 33.

Inasmuch as certain formal properties of the real number system are satisfied* by the preceding functions, it seems appropriate to call $\phi(x, y)$ the *pseudo-sum* of x and y; the function $\phi_1(x, y)$, the *pseudo-product* of x and y, and similarly for the functions $f(x, y)$ and $f_1(x, y)$. Moreover, to zero corresponds $f(x, x)$, a constant, and to unity corresponds $f_1(x, x)$, a constant.

We can, of course, further condition the functions $\phi(x, y)$, $f(x, y)$, etc., by referring to other properties of the real number system. For example, by assuming the relation

$$f(tx, ty) = tf(x, y) \tag{26}$$

in addition to the relation (7) we find that there exists a function $\chi(x)$ such that†

$$\chi\{t \cdot \chi^{-1}(x - y)\} = \chi\{t \cdot \chi^{-1}(x)\} - \chi\{t \cdot \chi^{-1}(y)\}. \tag{27}$$

Hence

$$\chi\{t \cdot \chi^{-1}(x)\} = ax, \tag{28}$$

where a is an arbitrary constant. From (28) we have

$$\chi^{-1}(ax) \cdot \chi^{-1}(1) = \chi^{-1}(a)\chi^{-1}(x). \tag{29}$$

Therefore

$$\chi^{-1}(x) = \chi^{-1}(1) \cdot x^c,$$

where c is an arbitrary constant. Similarly, if instead of (26) we assume

$$f(tx, ty) = f(x, y)$$

or

$$f(t + x, t + y) = f(x, y),$$

we find respectively the relations

$$\chi^{-1}\{\chi(tx) - \chi(ty)\} = \chi^{-1}\{\chi(x) - \chi(y)\}$$

or

$$\chi^{-1}\{\chi(t + x) - \chi(t + y)\} = \chi^{-1}\{\chi(x) - \chi(y)\},$$

which yield,‡ essentially, well known functional relations due to Cauchy.

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* Cf. also Dickson, *Transactions*, vol. 4 (1903), pp. 14, 17.

† Cf. Bourlet's "additive transmutation," l. c., pp. 136, 137, 141, etc.

‡ Cf. Bourlet, l. c., p. 143 (8).

SHORTER NOTICES.

Leçons sur le Calcul des Variations. Par J. HADAMARD. Recueillies par M. FRÉCHET. Tome Premier. Paris, Hermann, 1910. 8vo. viii+520 pp. 18 francs.

A NEW aspect is given to the calculus of variations as it is presented by Hadamard in his emphasis upon the functional character of the whole problem. While this aspect of the subject has been mentioned by several writers before Hadamard's treatise appeared, it is an innovation in a formal treatment of the entire subject. The ideas of the functional calculus do not appear on the surface of things in a manner offensive to the traditions, but they are subconsciously present throughout the book, as is explained by Hadamard himself in the preface; and they crop out at the surface occasionally.

Briefly stated, any line integral between fixed end points, for example, is a function of the path of integration, in the sense that when that path is given, the integral is determined. From this standpoint, a problem of the calculus of variations is a problem in finding the maximum or minimum values of a functional expression, the independent variable being a *curve.* Of course, we wish also to determine the value of the independent variable—that is the curve—for which the extreme value occurs.

Since the book is now well known to all who are interested in the calculus of variations, this review must be a retrospect and an appreciation rather than an analysis. Much as the reviewer regrets the obstacles which have prevented an earlier review, this changed viewpoint is not without advantages.

Hadamard at once commanded respectful attention everywhere; an immediate review would not have increased nor diminished the desire for instant examination of the work. The reception the book has received throughout the world fully justifies this sweeping statement. Thus Carathéodory in the *Bulletin des Sciences Mathématiques* (volume 34, pages 124–142) calls it an epoch-making work (un livre qui marquera une date dans l'histoire du sujet). I will content myself with noting briefly the plan of organization and what appear to be the guiding motives.

The book proper is preceded by an introduction on extrema of functions and on the properties of differential equations, and is followed by an appendix on implicit functions. These treatments are auxiliary and consider only questions necessary for the developments of the book itself.

The body of the work is divided into three main divisions, called books. In the first book, the task attempted is only the setting of the problem. This is done, to be sure, in a fashion not at all naïve. In fact, the difficulties peculiar to the subject, such as the familiar distinction between strong and weak extrema, the generalized notion of neighborhood, and so on, are fundamentally interesting from the standpoint of the functional calculus; every point of this character is discussed in full, and in a most interesting manner.

In the second book, the conditions of the first order—those associated commonly with the first variation—are discussed, together with generalizations of the problem. Here again, questions of peculiar fundamental difficulty are always discussed carefully as a consideration of the first moment. Thus the disagreement between the problem stated in homogeneous form and that stated in non-homogeneous form, which is not always clearly recognized, is made vividly clear. The preliminary Euler condition, the first variation, the fundamental lemma, conjugate points, transversals, and so on, are discussed, usually without great variation from methods now grown classical.

Especial emphasis is laid on practical applications, too often much neglected. Attention is called particularly to No. 140, page 151, and to No. 314, page 381, in which Hadamard sets forth the claim that the methods of mechanics and those of the treatment of geodetic lines by Darboux not only constitute a special case but practically forecast in a trustworthy manner the entire theory. Not only these broad claims but also profuse individual problems characterize the entire book as one of more than usual direct practical bearing.

In the later chapters of this second book are considered many generalizations of the first problem: the case of several variables, the case on a surface, the case of variable endpoints, the isoperimetric problem, and a detailed account of Mayer's problem.

This last—the problem of Mayer—is a second instance of a treatment of functional expressions, the first instance being the ordinary problem of the calculus of variations. It is used as a point of departure for a brief but illuminating chapter on the functional calculus, which is the real motive of the entire book, as explained above. The ideas of Volterra are first outlined, including the notion of a derivative of a functional expression. Then the concept of a linear functional expression is introduced, and some of the results of Hadamard, Fréchet and others are expounded. An interesting discussion of this part is given by Carathéodory in his review mentioned above.

The third book consists of a discussion of the final conditions of all types, for the general case of a "free" extremum. This discussion is more closely traditional, perhaps, though it is by no means slavish in following the methods of previous writers in detail. The second variation, the conditions of Jacobi, the fundamental Weierstrass theory, the methods of Hilbert and Kneser, are discussed fully; and such other problems as the case of discontinuous solutions (Carathéodory) and the existence of an absolute extremum (Osgood, Hilbert) are given in satisfactory completeness.

In all, the treatment is certainly well planned and well balanced. Due emphasis is given to generalizations, but the simplest forms of the problem predominate, and the treatment is therefore not inordinately complicated. The work of many authors is presented in a thoroughly digested form, and in a manner which is at once comprehensive and comprehensible; the student is given a well-rounded view of the entire subject. The special interests of the author are limited to their proper proportions as compared with the work of others.

Hadamard's treatise has already affected the development of the calculus of variations and that of the functional calculus; its influence on future developments should be profound.

E. R. HEDRICK.

Les Principes de l'Analyse mathématique. Par PIERRE BOUTROUX. Exposé historique et critique. Tome premier. Paris, Hermann, 1914. 8vo. xi + 547 pp.

THIS work is designed for those who desire a comprehensive view of mathematics, for the purpose of becoming acquainted with its intrinsic significance, and its historical evolution. The object of the book is to exhibit the facts of mathematics, rather than the methods of demonstration or discovery, from the most elementary to some of the most complex. It is intended to be serviceable to students of mathematics and equally to students of philosophy who desire to be armed with some knowledge of mathematics.

The fundamental basis of an enterprise of this kind is of course the first thing one wishes to find. This we do not discover explicitly stated in the tome under consideration, but in another place where three chapters of this work are printed under the title: "L'objet et la méthode de l'analyse mathématique"* the purpose becomes more evident, particularly as part of the article cited is from the second tome, not yet published. In a different place† the author states his view rather succinctly, in these terms: "The progress of analysis appears to us, not as a continuous evolution, . . . but as an endless duel in which the human mind, which seeks to subdue rebellious matter, gains more and more the advantage, thanks to its suppleness in continually modifying or enlarging its point of view and in fashioning new weapons." The mind meets a problem or class of problems, and whether it is able to solve them completely or not, it at least constructs an algebra of some kind, with which it is able to reach certain definite conclusions. Thus the unsolvability of algebraic equations of order above the fourth led to the theory of groups, which, though a most happy invention, for the discovery of the logical and algebraical nature of the connections between laws or mathematical functions, is, nevertheless, like the algebra of the sixteenth and seventeenth centuries, of limited power, neither the end nor the center of mathematical analysis. In like manner the theory of functions far from becoming elucidated as the years roll by, has become unfathomable, and we shall never succeed in getting it entirely within the limits of our science. We can set up a stable theory of functions only by restricting (and in an arbitrary way for the most part) our conception of functional correspondence.

We find accordingly that the two books of this tome are called respectively, Statement of the facts, and Construction. Under the first heading one finds: numbers; magnitudes, figures, and combinatory calculus. Under the second are grouped algebraic calculus, calculus of functions, and algebraic geometry. We are told that the next tome will contain analytic geometry, imaginary quantities, series, logic of mathematics, and infinitesimal calculus. In the reference* given we find the reason for these divisions, as follows. The scientist was at first content to gather facts. He looked around him and perceived the harmonious properties of the world of numbers and the world of geometric figures, and also of measurable magnitudes, the synthesis of quantity and figure, the union of arithmetic and geometry. But with the diffusion of algebra he ceased to be contemplative, and became constructive. From simple pieces he built more and more elaborate designs until he became so charmed with his creations that he left to the physicist entirely the application and interpretation of his theories. In the end his principal aim ceased to be the acquisition of new facts but became the perfecting of methods and processes. The ideal of this phase of mathematics is the algebraico-logical synthesis, by which one would start from a few premises and deduce with no further reference to the world of facts the whole of mathematics. In this attempt however the mathematician is turned back upon himself, and his work begins to consist in the more minute analysis of his notions and the discernment of subtilities that previously had escaped his attention. Thus we pass from the algebraico-logical synthesis to analysis in the proper sense of the term. For example, we no longer look upon an ellipse as made up of an infinity of infinitesimal particles called points. An ellipse is a law, the law of a locus. This law is pregnant with the properties of the ellipse and a sufficient analysis of it will bring out all the characteristic properties of the ellipse. From this point of view we find the tangent and the curvature, or the area and the length. This is what we find in the infinitesimal calculus when we leave out the infinity. But the object of the analysis is ever new construction.

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* *Revue de Métaphysique et de Morale*, May, 1913.

† Same, January, 1913.

"Having mined the simple notions like those of algebraic relations or the classical geometric figures, the mathematician attacks the fundamental principles upon which repose the algebraico-logical construction.

"He seeks for example, how one may present and particularize the axioms of geometry in order to simplify so far as possible those he is forced to admit without demonstration, and to reduce their number to a minimum.

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* *Revue de Métaphysique et de Morale*, May, 1913.

"He analyzes the notion of continuity in order to discover what is essential in it for his purposes, and what hypotheses it is necessary but sufficient to make, regarding an aggregate that is discontinuous, whether of points or of numbers, in order to be able to reason about this aggregate as he does about the continuum.

"Then penetrating to the heart of mathematics, the analyst attacks the general notion of function (law of correspondence between magnitudes) such as he conceives it to be a priori. He searches out how he ought to determine this notion, and in what measure it must be limited, in order to subject the function to his methods of calculation and to represent it quantitatively by algebraic expressions. He sees thus the possibility of generalizing and extending considerably the algebraic technique of his predecessors."

The first tome of this interesting work is pleasant reading. It is quite elementary on the whole and the references to parts of the next tome lead one to suppose that some of the more philosophical notions will appear there. There are some inaccuracies noticeable. For example we remark the misuse of the term *associativity* with regard to involution, page 11. Division by zero, page 48, is left a little hazy as to whether the sign ∞ is intended to be included as a number or not. The term *rationnel* in the last note on page 81, does not seem to be what the author intends to say. On page 137 the series for π as well as the decimal, are inaccurate. On page 138 at the bottom, the 2 in front of each radical should be omitted. On page 139 the continued fraction represents $4/\pi$. The figure on page 249 is badly drawn. On page 331 the formula for the second root, x'', needs a 4. However these are slips, and we need not dwell on them. The fundamentals that come out here and there are the vital part. Such as, page 44, "The world of numbers is for us essentially a class of abstract elements, about which we suppose nothing, save that they are subject to certain definite operations." On page 145, "Every relative number is defined by an ordinary number (rational or irrational, called the absolute value of the relative number) and by a sign + or −." On page 275, "Algebra is a rule, or more exactly, the totality of rules, according to which one carries out certain transformations or combinations called algebraic; in general these combinations are those defined by the fundamental operations of arithmetic, but nothing prevents our imagining others." We have however probably said sufficient to show that there is an interest in the book for everyone. JAMES BYRNIE SHAW.

Principes de la Théorie des Fonctions entières d'Ordre infini. By OTTO BLUMENTHAL. Paris, Gauthier-Villars, 1910. vi + 147 pp.

Entire functions of finite order as well as certain classes of entire functions of infinite order have been treated by a number of mathematicians in recent years. In this volume of the Borel series of monographs on the theory of functions, Blumenthal considers the *general* entire function of infinite order, so that the book forms a natural and satisfactory sequel to Borel's own Leçons sur les Fonctions entières of the same series. The interest of the results obtained lies in their generality rather than in their applicability to special entire functions not before treated. These results are in large measure original with Blumenthal although a similar range of ideas had been earlier developed by Kraft (Dissertation, Göttingen, 1903).

It is by the aid of the notion of function-type (fonction-type) that Blumenthal is enabled to overcome the inherent difficulties of the problems which arise. Let $\nu(x)$ be a function which increases to $+\infty$ with x, and let $\mu(x) \geqq \nu(x)$ be a like function whose rate of increase is governed by an inequality

$$\mu(x') \leqq \mu(x)^{1+\epsilon(x)}, \quad x' = x^{1+\frac{1}{\mu(x)}\epsilon(x)}$$

where $\epsilon(x)$ is a decreasing infinitesimal. Then $\mu(x)$ is a *function-type adjoint to* $\nu(x)$ *and* $\epsilon(x)$ if the inequality

$$\mu(x) \leqq \nu(x)^{1+\delta}$$

holds for any $\delta > 0$ and an infinite number of values of x. If $\nu(x)$ is given it is clear that $\mu(x)$ yields a measure of the increase of $\nu(x)$, and at the same time possesses a certain regularity of increase (croissance typique).

The fundamental theorem concerning function-types is that corresponding to any given $\nu(x)$ a function-type $\mu(x)$ adjoint to $\nu(x)$ and some infinitesimal $\epsilon(x)$ may be found. The proof first given by Blumenthal (pages 24–31) contains an error. It is assumed that for any given increasing function $w(x)$ an infinitesimal $\epsilon(x)$ (not the $\epsilon(x)$ of the theorem) can be found such that both $\epsilon(x) \log x$ and $\epsilon(x) \log w(x)$ are increasing functions. But if $\log \log w(x)$ is made to increase by only a finite amount outside of a set of intervals $(x_i, x_i + \delta_i)$, $i = 1, 2, \cdots$, and if these intervals be so chosen that the series

$$\frac{\delta_1}{x_1} + \frac{\delta_2}{x_2} + \cdots$$

converges, it is easy to see that $\epsilon(x)$ cannot approach zero. The second proof (note I, pages 117–128) is however devoid of any objection.

With the aid of the concept of function-type Blumenthal generalizes almost all the results known for entire functions of finite order but it is perhaps not desirable that the reviewer make an outline of this material.

Here then is a book which the mathematician who is interested in the theory of the entire function will find worthy of his attention. GEORGE D. BIRKHOFF.

Les Systèmes d'Equations linéaires a une Infinité d'Inconnues. Par FRÉDÉRIC RIESZ. Paris, Gauthier-Villars, 1913. vi + 182 pages. 6.50 fr.

THIS little book belongs to the collection of monographs on the theory of functions published under the general direction of M. Emile Borel. It deserves the high praise of being pronounced worthy a place in this excellent series.

The purpose of the volume is to give a rapid exposition of the fundamental ideas, of the methods and of the principal results in the theory of linear equations with an infinite number of variables—a theory which is due almost entirely to contemporary mathematicians.

An introduction to the subject is made through a chapter (Chapter I, pages 1–20) devoted to the beginnings of the theory. The method of undetermined coefficients was the first; but this method is not characteristic of the subject. Next comes the work of Fourier, who introduced an important general principle in connection with a special type of problem: naturally, the work of Fourier does not meet the modern requirements of rigor. Mention is also made of the papers of Fürstenau and Kötteritzsch; these are said to have been without importance in the development of the theory. (In this connection see a paper by the reviewer in the *American Journal of Mathematics* for January, 1914.) Finally, an account of the work of Appell is given in the latter part of the first chapter.

Chapter II (pages 21–41) begins with a brief account of the ideas introduced by the American astronomer, Dr. G. W. Hill, in his very important memoir on the motion of the lunar perigee. This memoir was the starting point for the modern theory of infinite determinants and of linear equations with an infinite number of variables. Poincaré first supplied the convergence proofs which are needed to justify the processes used formally by Hill. With the ideas of Hill and Poincaré as a basis, a theory of considerable extent and wide range of applicability has been built up, principally by H. von Koch. An account of the general elements of this theory, in their simplest form and without applications, is given in Chapter II.

Up to this point the author has followed the chronological order, except that (in Chapter I) he has given an account of the work of Appell before that of Hill. He now inverts the chronological order, for greater convenience in exposition, giving an account (in Chapter III, pages 42–77) of Schmidt's far-reaching theory before taking up the earlier work of Hilbert. This theory of Schmidt, the most general hitherto developed, is itself extended by the present author so as to include a somewhat larger range of results than those obtained by Schmidt in his original account.

In connection with this chapter one would do well to read the elegant paper by Bôcher and Brand in the *Annals of Mathematics* (2), 13 (1912): 167–186, where an exposition of the Schmidt theory is given which leaves nothing to be desired from the point of view of the reader who is forming a first acquaintance with the subject. In order to have a deep understanding of the theory the reader should also see the geometric interpretation given to the whole matter in the original presentation by Schmidt. This latter paper is fundamentally so simple and elegant that it is sure to become one of the classics of mathematical literature. It is to be regretted that the book under review does not contain an exposition of the geometric ideas lying at the basis of this theory of Schmidt. By means of these geometric considerations the whole matter receives an illumination which probably can be procured for it in no other way.

Chapters IV (pages 78–121) and V (pages 122–155) are devoted to the theory of Hilbert as expounded by him in his now famous memoirs on the theory of integral equations and by some of his disciples in subsequent contributions. The first of these two chapters is given to the theory of linear substitutions where the number of variables is infinite. The general theory which is developed from this point of view is applied to the derivation of important results concerning a certain type of system of equations with an infinite number of variables. The second of these two chapters is given to the theory of quadratic forms where the number of variables is infinite and to the application of this theory to that of linear equations.

Chapter VI (pages 156–180) is devoted to certain applications of the general theory developed in the preceding part of the book. It falls into three parts dealing with as many distinct topics; namely, linear differential equations in which the coefficients are expansible in Laurent series, integral equations and trigonometric series.

This little book will serve a useful end by affording a ready introduction to one of the most important and most readily accessible phases of the general theory of functions of an infinite number of variables, a field in which at present there lies out before us a vast domain of unexplored territory—a domain in which the present generation will probably make further important explorations.

R. D. CARMICHAEL.

A General Course of Pure Mathematics from Indices to Solid Analytical Geometry. By ARTHUR L. BOWLEY, Sc.D. Oxford, Clarendon Press, 1913. xii + 272 pp.

It would be difficult to give a better brief account of the contents and the purpose of this book than that supplied by the author in the preface. From his remarks, therefore, we shall quote a few sentences, as follows:

"This book is the result of an attempt to bring within two covers a wide region of pure mathematics. Knowledge is assumed of that part of mathematics usually required for matriculation, namely algebra to simultaneous quadratic equations and the substance of the first four books of Euclid, together with a very slight acquaintance with graphic algebra, mensuration, and solid geometry. From this stage the work is carried forward in algebra to the logarithmic series; in coordinate geometry to the nature of the general conicoid; in trigonometry to the use of Euler's expressions for the sine and cosine, with a careful treatment of imaginary quantities; in calculus to definite integration and the maxima of a function of n independent variables; together with the pure geometry which is necessary for the other subjects. It has been the intention to include the bulk of the results obtained in pure mathematics which admit of rigid proof of a fairly easy character, and are needed by those who use pure mathematics as an instrument in mechanics, engineering, physics, chemistry, and economics. For this purpose a very great deal that is ordinarily contained in text-books has been thrown aside, and only those theorems and formulas which are of direct practical application or which are necessary to lead to others of direct practical application are retained.

"It has also been the intention to give exact definitions and strict proofs, of a more careful nature than those found in many of the more diffuse and elementary books; only two difficulties have been intentionally glozed over, viz., the nature of continuity and the nature of irrationals."

The considerable number of topics, the discussion of which is brought together in this one book, are treated in separate sections and probably in a larger degree of isolation from each other than most readers would expect in a single volume in which all of them find a place. The exposition, on the whole, is fairly satisfactory; some of the sections are excellent. The section on limits and series is the least satisfactory of all; some of the statements in it are properly characterized as awkward. For examples of these awkward statements the reader may see pages 101, 102, 105, 115.

The book as a whole is a contribution of some value to the pedagogy of that part of the mathematical curriculum with which it is concerned. Some of the controlling ideas in its preparation might well be adapted to the needs of American institutions; but the book itself is probably not well suited to such purposes. R. D. CARMICHAEL.

Démonstration du Théorème de Fermat. Par E. FABRY. Paris, Hermann et Fils, 1913. 22 pp. 1.50 fr.

Using the theory of Kummer as a basis the author undertakes to prove that the equation

$$x^\lambda + y^\lambda + z^\lambda = 0,$$

where λ is an odd prime, cannot be true for any three integers x, y, z prime each to each, provided that

(1) no one of these integers is divisible by λ;

(2) one of them is divisible by λ but not by λ^2.

In the discussion of the first of these results there occurs an essential error which has already been pointed out by Mirimanoff (*Comptes rendus*, Paris, 157: 491–492). Fabry's second result was already known to Sophie Germain and Legendre (see Bachmann's Niedere Zahlentheorie, II, page 467).

R. D. CARMICHAEL.

Vectorial Mechanics. By L. SILBERSTEIN. Macmillan and Company, London, 1913. viii + 197 pp.

THERE are not in English so many books on vector analysis and its applications that we may not welcome another. The Gibbs-Wilson is the most extended and detailed as regards vector analysis itself, but contains illustrations from geometry, mechanics, and physics rather than applications to them; it has therefore too much mathematics and too little connected application to be entirely ideal for the young physicist. Coffin's is more evenly balanced, and may serve almost equally well as an introduction to vector analysis and to vector physics. Heaviside's genial treatment is embedded in his Electromagnetic Theory. Now comes Silberstein with a work which passes as lightly as possible over formal vector analysis and concentrates on theoretical mechanics. This is a useful variety to introduce. The notation is that of Heaviside; heavy type for vectors, no sign for the scalar product; and a prefixed V for the vector product.

A number of minor complaints may well be made. The Macmillan zero, more insignificant than an "o," is bad. It is unfortunate to use Clarendons for lettering a figure, especially when italics are used in the text. And what can be the advantage of making the figures (usually) run with their positive direction clockwise as they appear on the page? Why take the velocity potential φ so that $\mathbf{v} = \nabla\varphi$ instead of $\mathbf{v} = -\nabla\varphi$, especially now that Lamb in his classic Hydrodynamics has decided in favor of the latter choice? There are a number of instances which show that the author, despite varied linguistic accomplishments, does not sense the meaning of common English words—otherwise he would not call the component of a vector a scalar, nor would he speak of the algebraic product of the tensors of two vectors after defining the tensor as a positive number. And when he speaks of "that $\Delta\sigma$ of which $d\sigma$ is the limit" he is probably more unfortunate than wrong. Indeed the meaning of his text is seldom in question.

The one very bad mistake we have discovered is the treatment of the curl and of Stokes's theorem (pages 33–4). Silberstein attempts an intrinsic proof by considering

$$\psi = \lim_{\Delta S = 0} \frac{\int_0 \mathbf{f} \bullet d\mathbf{r}}{\Delta S}.$$

The scalar function ψ depends on position x, y, z and on α, β, γ, the direction cosines of the normal to a planar element at x, y, z. He then says that ψ may be written as the component of a vector **C** along the normal **n**, $\psi = \mathbf{C} \bullet \mathbf{n}$. That this does not follow is patent. Indeed if we assume that the limit ψ exists, the only hard thing about Stokes's theorem is precisely to show that ψ may be written as $\mathbf{C} \bullet \mathbf{n}$ with **C** dependent on x, y, z but not on α, β, γ. I treated the intrinsic method in an article "Divergence and curl," *American Journal of Science*, volume 23 (1907), pages 214–220, and it has probably been treated by others. We may add that the author uses curl, infinitesimal rotation, and other related ideas so often that before his book is finished he has more than enough material to form a proof of Stokes's theorem.

The first quarter of the Vectorial Mechanics develops the elements of vector analysis, including differentiation and integration, but with no reference to the linear vector function. The student would profit by having at hand a more detailed treatment; but it would be difficult for him to find one where the author seemed so thoroughly to think vectors. The fundamental definition of a vector as a free vector and the relation of the definition to directed quantities in physics is explained fairly well, much better than by Coffin. No distinction is made between axial and polar vector, or between scalars and pseudo-scalars—for which we may be thankful.

Mechanics starts off with d'Alembert's principle, Lagrange's equations, and Hamilton's principle. From this it may be inferred that the reader should have considerable familiarity with mechanics before commencing this book. Next we find the principles of work and energy, center of gravity (momentum), and angular momentum (areas). The treatment of the motion of a rigid body is very pretty. It is, however, necessary to introduce the notion of the symmetric linear function, and unless the reader looks up the reference to Heaviside or similar source, he may find the chapter a trifle hard. The Poinsot motion yields with great ease to vector methods, and the author makes the most of his analysis.

The chapter on the mechanics of deformable bodies begins with a concise summary of properties of the general linear vector function, passes to strains and infinitesimal displacements, discusses surfaces of discontinuity (Hadamard), and terminates with stress. It would be difficult in so short a space (47 pages) to do the work better. There follows thirty pages on hydrodynamics which contain most of the classical theory as far as general properties are concerned, and a few other things. This, too, is thoroughly good.

To this point we have covered only 170 pages of the text. The brevity is partly a matter of conciseness in style, but largely due to systematically thinking and using vectors. Sixty-nine exercises, a table of cartesian-vector equivalents in parallel column, and an index complete the work. We could only wish for fifty pages more in which the classical electromagnetic theory, including a little crystal optics, should be presented as succinctly as the theory of rigid motion, fluid motion, and elastic media. As it is, however, Silberstein has given us an almost ideal introduction to mathematical vector physics.

EDWIN B. WILSON.

NOTES.

THE July number (volume 15, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "A new principle in the geometry of numbers, with some applications," by H. F. BLICHFELDT; "An application of Severi's theory of a basis to the Kummer and Weddle surfaces," by F. R. SHARPE and C. F. CRAIG; "Transformations of surfaces of Voss," by L. P. EISENHART; "Birational transformations of certain quartic surfaces," by F. R. SHARPE and VIRGIL SNYDER; "One-parameter families of curves in the plane," by G. M. GREEN; "The minimum of a definite integral for unilateral variations in space," by G. A. BLISS and A. L. UNDERHILL; "On a method of comparison for triple-systems," by L. D. CUMMINGS; "An existence theorem for a certain differential equation of the nth order," by W. R. LONGLEY.

THE July number (volume 36, number 3) of the *American Journal of Mathematics* contains the following papers: "On a certain completely integrable system of linear partial differential equations," by E. J. WILCZYNSKI; "On the connection of an abstract set, with applications to the theory of functions of a general variable," by A. D. PITCHER; "On series of iterated linear fractional functions," by R. D. CARMICHAEL; "The derivative of a function of a surface," by C. A. FISCHER; "Some invariants and covariants of ternary collineations," by H. B. PHILLIPS; "A geometrical application of the theory of the binary quintic," by FLORENCE P. LEWIS.

THE Prince Jablonowski society of Leipzig announces the following prize problem for the year 1916: "To extend the theory of linear functional differential equations in any direction. Particularly desirable is a complete consideration of new special cases." The value of the prize is 1,050 marks. Competing memoirs should be in the hands of the secretary before October 31, 1916.

THE academy of sciences of Paris has awarded its Poncelet prize for 1914 to Dr. H. LEBESGUE, of the University of Paris. The grand prize was not awarded. The Lalande prize in theoretical astronomy was awarded to Dr. M. GUILLAUME of the University of Lyon; the Valz prize was divided between Dr. P. CHEVALIER of Shanghai, and Dr. P. SALET, of the observatory of Paris.

THE following doctorates in mathematics and astronomy were conferred by the French universities during the academic year 1912–1913. A. BLONDEL; "On the theory of tides in a canal; application to the Red Sea" (Toulouse); M. LUIZET; "The Cepheides considered as double stars, with a chart of δ" (Lyon); K. POPOFF: "On the movement of 108 Hecube" (Paris); A. VÉRONNET: "Rotation of a heterogeneous ellipsoid and exact figure of the earth" (Paris); P. J. BROWNE: "The problem of inversion proposed by Abel and its generalizations" (Toulouse); G. COTTY: "Abelian functions and the theory of numbers" (Toulouse); G. FASBINDER: "Dynamics of variable systems and the rotation of the earth" (Paris); M. GEVREY: "Partial differential equations of parabolic type" (Paris); C. PLATRIER: "The minors of Fredholm's determinant and systems of linear integral equations" (Paris); L. ROCHE: "The wave surface in rotative magnetic polarization and more general phenomena" (Paris); L. ROUYER: "The deformation of quadrics and conjugate surfaces with regard to a quadratic complex" (Paris); J. TROUSSET: "Semi-analytic study of the movement of the eighth satellite of Jupiter" (Paris).

IN memory of HENRI POINCARÉ an international committee has been formed for the purpose of raising a subscription, the proceeds of which, after striking a medal of Poincaré, will be administered by the Paris Academy in encouraging young scientists engaged in the fields to which Poincaré contributed especially, viz., analysis, celestial mechanics, mathematical physics, scientific philosophy. Contributors of 25 to 50 francs will receive a bronze medal, of 50 francs or more a silver medal. Subscriptions should be sent to M. E. Lebon, secretary-treasurer, rue des Ecoles, 4 bis, Paris.

THE following advanced courses in mathematics (elementary courses not included) are offered at the Italian universities during the academic year 1914–1915:

UNIVERSITY OF BOLOGNA.—By Professor P. BURGATTI: Theory of elasticity, especially elastic vibrations, three hours. —By Professor L. DONATI: Electrodynamics of moving bodies; thermodynamics; radiations; hypothesis of quanta with applications, three hours.—By Professor F. ENRIQUES: Theory of algebraic curves and surfaces, three hours.—By Professor S. PINCHERLE: Elliptic functions; integral equations and systems of linear equations with infinite unknowns, three hours.

UNIVERSITY OF CATANIA.—By Professor E. DANIELE: Equilibrium of elastic bodies, four hours.—By Professor M. DE FRANCHIS: Geometry on algebraic surfaces by transcendental aids, four hours.—By Professor G. PENNACCHIETTI: Hydrodynamics, four hours.—By Professor C. SEVERINI: Theory of analytic functions; theory of permutable functions, four hours.

UNIVERSITY OF GENOA.—By Professor E. E. LEVI: Calculus of variations, four hours.—By Professor G. LORIA: Geometric applications of elliptic functions, three hours.—By Professor O. TEDONE: Geometrical and physical optics, three hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics; the ancient ages, three hours.—By Professor A. DEL RE: n-dimensional analysis of Grassmann with applications to mechanics of the spaces of constant curvature, four and one half hours.—By Professor R. MARCOLONGO: Equations of dynamics; periodic solutions; asymptotic solutions; restricted problem of the three bodies, three hours.—By Professor D. MONTESANO: Theory of algebraic surfaces and of their linear systems; theory of birational transformations of three-dimensional space, three hours.—By Professor E. PASCAL: The functions of lines and the calculus of variations, three hours.—By Professor L. PINTO: Thermodynamics, three hours.—By Professor G. TORELLI: One—and more—dimensional sets; simple and multiple Lebesgue's integrals; functions of a set; derivation of indefinite integrals, four and one half hours.

UNIVERSITY OF PADUA.—By Professor F. D'ARCAIS: Functions of a complex variable; integral equations, four hours.—By Professor A. COMESSATTI: Algebraic geometry, three hours.—By Professor P. GARRANIGA: Theory of numbers, three hours.—By Professor T. LEVI-CIVITA: Analytic dynamics; problem of the three bodies, four and one half hours.—By Professor G. RICCI: Absolute differential calculus; potential; elasticity, four hours.—By Professor F. SEVERI: Linear systems of plane curves and rational surfaces, four hours.—By Professor A. TONOLO: Fourier's series; partial differential equations, three hours.—By Professor G. VERONESE: Geometrical applications of the theory of sets, four hours.

UNIVERSITY OF PALERMO.—By Professor G. BAGNERA: Functions of a complex variable; entire functions of one or of two variables, three hours.—By Professor M. GEBBIA: Electrodynamics (advanced part), four and one half hours.—By Professor G. B. GUCCIA: General theory of algebraic curves and surfaces, four and one half hours.—By Professor A. VENTURI: Foundations of the modern methods in celestial mechanics following Poincaré; method of Hill for the moon, four and one half hours.

UNIVERSITY OF PAVIA.—By Professor L. BERZOLARI: Geometry on an algebraic curve, and applications to linear systems of plane algebraic curves, three hours.—By Professor E. BOMPIANI: Differential geometry, three hours.—By Professor U. CISOTTI: Mechanics of continuous systems; theory of potential; electricity, three hours.—By Professor F. GERBALDI: Functions of a complex variable; elliptic functions, three hours.—By Professor G. VIVANTI: Integral equations, three hours.

UNIVERSITY OF PISA.—By Professor E. BERTINI: Cremona transformations in the plane and in space, three hours.—By Professor G. BIANCHI: Functions of a complex variable; linear differential equations, four and one half hours.—By Professor U. DINI: Integral equations; linear differential equations in the real field, four and one half hours.—By Professor G. A. MAGGI: Principles of analytic mechanics; theory of potential functions; phenomenological theory of electromagnetic field, four and one half hours.—By Professor P. PIZZETTI: Theory of interpolation; general notions of spherical astronomy; general theory of perturbations, four and one half hours.

UNIVERSITY OF ROME.—By Professor G. BISCONCINI: Geometrical applications of calculus, three hours.—By Professor G. CASTELNUOVO: Calculus of probability, three hours.—By Professor L. SILLA: Kinematics and mechanisms, three hours. —By Professor V. VOLTERRA: Permutable functions; functional derivative equations with applications, three hours.—Elasticity, three hours.—By ————: Theory of functions; elliptic functions, three hours.

UNIVERSITY OF TURIN.—By Professor T. BOGGIO: Potential functions and hydrodynamics, three hours.—By Professor G. FUBINI: Calculus of variations; Fourier's series; the principle of minimum as application of Fourier's series to the calculus of variations, three hours.—By Professor C. SEGRE: Theory of invariants applied to geometry, three hours.—By Professor C. SOMIGLIANA: Magnetism and electromagnetism, three hours.

THE London mathematical society has awarded the de Morgan medal for 1914 to Professor Sir JOSEPH LARMOR, of Cambridge University.

Dr. A. Bohr, of the University of Copenhagen, has been appointed reader in mathematical physics at the University of Manchester.

Dr. U. Crudeli has been appointed docent in mathematical physics at the University of Rome.

Professor L. Tonelli, of the University of Cagliari, will occupy the chair of analysis at the University of Parma during the present year.

Dr. E. Bompiani has been appointed docent in analytic geometry at the University of Pavia.

Dr. A. Comessatti has been appointed docent in geometry at the University of Padua.

Dr. E. Rosati has been appointed docent in projective geometry at the University of Pisa.

Dr. M. Picone has been appointed docent in mathematical analysis at the University of Turin.

Dr. A. Tonolo has been appointed docent in mathematical analysis at the University of Padua.

Professors G. Scheffers and C. Jolles, of the technical school of Berlin have been named Geheime Regierungs-Räte by the German emperor.

Professor A. Gutzmer has been elected rector of the University of Halle for the coming academic year.

Professor A. Korn has been appointed honorary professor at the Charlottenburg technical school.

Dr. E. Hellinger, of the University of Marburg, has been appointed associate professor of mathematics at the University of Frankfort.

Professor P. Stäckel, of the University of Heidelberg, has been elected honorary member of the mathematical-physical society of Budapest.

PROFESSOR D. K. PICKEN, of Victoria College, New Zealand, has been appointed master of Ormand College, University of Melbourne.

THE University of Groningen conferred the honorary degree of doctor of mathematics and physics on Professor E. B. VAN VLECK at its recent tercentenary celebration.

PROFESSOR L. E. DICKSON, of the University of Chicago, will lecture at the University of California during the first term of the present academic year.

PROFESSOR ELIJAH SWIFT, of Princeton University, has been appointed Williams professor of mathematics at the University of Vermont.

DR. M. O. TRIPP has been appointed professor of mathematics at Olivet College.

DR. J. E. ROWE, of Dartmouth College, has been appointed assistant professor of mathematics in the Pennsylvania State College.

PROFESSOR W. C. KRATHWOHL, of Ripon College, has been appointed assistant professor of mathematics in the Armour Institute of Technology.

AT the College of the City of New York, Dr. E. E. WHITFORD has been promoted to an assistant professorship of mathematics. Messrs. G. M. HAYES, C. A. TOUSSAINT, J. A. BREWSTER, and S. A. SCHWARZ have been promoted to instructorships in mathematics.

AT Cornell University, Dr. W. A. HURWITZ has been promoted to an assistant professorship of mathematics.

DR. K. P. WILLIAMS, of Indiana University, has been promoted to an assistant professorship of mathematics.

MR. J. D. ESHLEMAN has been appointed instructor in mathematics at Western Reserve University.

DR. L. E. WILLIAMS has been appointed instructor in mathematics at the Georgia School of Technology.

MISS M. E. WELLS, of Mount Holyoke College, has been appointed instructor in mathematics at Oberlin College; Professor Mary E. SINCLAIR, of Oberlin, is absent on leave during the present academic year.

DR. SUSAN R. BENEDICT has been appointed associate professor of mathematics at Smith College.

AT the University of Minnesota, Dr. W. H. BUSSEY has been promoted to an associate professorship of mathematics. Dr. W. F. HOLMAN and Dr. H. L. SLOBIN have been promoted to assistant professorships of mathematics.

PROFESSOR E. P. R. DUVAL has returned to his former position as associate professor of mathematics in the University of Oklahoma.

AT Columbia University, Professor D. E. SMITH will be abroad on leave of absence during the coming academic year.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BORGOGELLI (G.). Di un più semplice modo per disegnare le proiezioni dei cinque poliedri regolari. 2a edizione. Roma, Manuzio, 1914. 16mo. 27 pp.

BUDDE (E.). Tensoren und Dyaden im dreidimensionalen Raum. Braunschweig, Vieweg, 1914. M. 6.80

DICKSON (L. E.). Elementary theory of equations. New York, Wiley, 1914. 8vo. 5+184 pp. Cloth. $1.75

——. Linear Algebras. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 16.) Cambridge, University Press, 1914. 8vo. 8 + 73 pp. 3s.

DUCLA (V.). Démonstration d'un théorème de Fermat. 2e édition, revue et corrigée. Pau, Garet & Haristoy, 1914. 8vo. 30 pp.

DYCK (W. v.). Ueber den Verlauf der Integralkurven einer homogenen Differentialgleichung 1ter Ordnung. (*Abhandlungen der k. bayerischen Akademie der Wissenschaften.*) Munchen, Franz, 1914. 8vo. 49 pp. M. 2.00

ELLIOTT (C.). Models to illustrate the foundations of mathematics. Edinburgh, Lindsay, 1914. 8vo. 8+116 pp. 2s. 6d.

ENGEL (F.). See LIEBMANN (H.).

GODFREY (C.) and SIDDONS (A. W.). First steps in the calculus. Cambridge, University Press, 1914. 8vo. 102 pp. Cloth. 1s. 6d.

HACK (F.). Wahrscheinlichkeitsrechnung. Neudruck. (Sammlung Goschen. Nr. 508.) Berlin, Göschen, 1914. 12mo. 123 pp. Cloth. M. 0.90

HELM (G.). Die Grundlehren der hoheren Mathematik. Neue verbesserte Ausgabe. Leipzig, Akademische Verlagsgesellschaft, 1914. 8vo. 15+419 pp. Cloth. M. 15.00

JUNKER (F.). Hohere Analysis. 2ter Teil: Integralrechnung. 3te, verbesserte Auflage. Neudruck. (Sammlung Göschen. Nr. 88.) Berlin, Goschen, 1914. 12mo. 190 pp. Cloth. M. 0.90

KÖNIG (J.). Neue Grundlagen der Logik, Arithmetik und Mengenlehre, mit dem Bildnis des Verfassers. Leipzig, Veit, 1914. 8vo. 259 pp. M. 9.00

LAMPE (E.). Mathematik. (Aus "Deutschland unter Kaiser Wilhelm II.") Berlin, Hobbing, 1914.

LIEBMANN (H.) und ENGEL (F.). Die Berührungstransformationen. Geschichte und Invariantentheorie. Zwei Referate der Deutschen Mathematiker-Vereinigung. Leipzig, Teubner, 1914. 8vo. M. 3.00

LINDEMANN (F. und L.). See POINCARÉ (H.).

MACALESTER LOUP (W.). En bijdrage tot de theorie der cyclische oppervlakken en congruenties. (Thèse.) Utrecht, J. van Boekhoven, 1914. 4to. 80 pp.

MORITZ (R. E.). Memorabilia mathematica. The philomath's quotation-book. New York, Macmillan, 1914. 8vo. 9+410 pp. Cloth. $3.00

POINCARÉ (H.). Wissenschaft und Methode. Autorisierte deutsche Ausgabe mit erlauternden Anmerkungen von F. und L. Lindemann. (Wissenschaft und Hypothese. XVII.) Leipzig, Teubner, 1914. 8vo. 6+283 pp. M. 5.00

RAECKE (C.). Allgemeiner Beweis des letzten Fermatschen Satzes. Dresden, Becker, 1914.

REY (J.). Discurso leido en la solemne apertura del curso académico de 1913 à 1914. Oviedo, 1913.

SIDDONS (A. W.). See GODFREY (C.).

SISAM (C. H.). See SNYDER (V.).

SNYDER (V.) and SISAM (C. H.). Analytic geometry of space. New York, Holt, 1914. 12mo. 11 + 289 pp. $2.50

STUDY (E.). Die realistische Weltansicht und die Lehre vom Raume. Geometrie, Anschauung und Erfahrung. (*Die Wissenschaft*, Band 54.) Braunschweig, Vieweg, 1914. 8vo. M. 5.20

TIMERDING (H. E.). See WISSENSCHAFTEN.

VOSS (A.). See WISSENSCHAFTEN.

WILDBRETT (A.). Analytische und projektive Geometrie. 1ter Teil: Analytische Geometrie der Geraden und des Kreises. Elemente der projektiven Geometrie. Nurnberg, Korn, 1914. 8vo. 7+144 pp. Cloth. M. 2.90

Wissenschaften, Die mathematischen. (Die Kultur der Gegenwart, Teil III, Abteilung 1.) Lieferung II: A. Voss, Die Beziehungen der Mathematik zur Kultur der Gegenwart; H. E. Timerding, Die Verbreitung mathematischen Wissens und mathematischer Auffassung. Leipzig, Teubner, 1914. 8vo. 6+161 pp.

II. ELEMENTARY MATHEMATICS.

Barnard (S.) and Child (J. M.). A new algebra. Parts 5 and 6. With answers. London, Macmillan, 1914. 8vo. 276 pp. Cloth. 3s.

Borchardt (W. G.) and Perrott (A. D.). A junior trigonometry. With answers. London, Bell, 1914. 8vo. 258 pp. Cloth. 3s. 6d.

Brunn (J.). Vierstellige Logarithmen. 2te Auflage. Munster, Aschendorff, 1914. 8vo. 11 pp. M. 0.25

Camman (P.) et Rébouis (A. G.). Compléments de géométrie. Paris, Gigord, 1914. 8vo. 8+463 pp.

Cavazzoni (L.) e Cercignani (E.). Libro di geometria del piano. 2a edizione. Milano, Albrighi & Segati, 1914. 8vo. 300 pp. L. 2.80

Cercignani (E.). See Cavazzoni (L.).

Child (J. M.). See Barnard (S.).

Dalle (A.) et Waele (C. De). Géométrie plane et éléments de topographie. 4e édition. Namur, Wesmael-Charlier, 1914. 2+372 pp. Cloth. Fr. 2.75

Deckert (A.). Planimetrie, Stereometrie und Trigonometrie. Breslau, Goerlich, 1914. 8vo. 4+214 pp. M. 3.50

Diesener (H.). Die Buchstabenrechnung und Algebra. 4te umgearbeitete Auflage von E. Novaczkiewicz. Leipzig, Voigt, 1914. 8vo. 6+275 pp. M. 6.00

Dobbs (W. J.). A school course in geometry, including the elements of trigonometry and mensuration, and an introduction to the methods of co-ordinate geometry. With answers. London, Longmans, 1914. 8vo. 22+427 pp. 4s.

Fenkner (H.). Mathematisches Uebungsbuch. Ausgabe B. 1ter Teil. Pensum der Quarta, Tertia und Untersekunda. Berlin, Salle, 1914. 8vo. 4+376 pp. Cloth. M. 3.40

Friedrich (M.). Grundzuge der analytischen Geometrie. 3te Auflage, durchgesehen und verbessert von G. Ehrig. Leipzig, Weber, 1914. 8vo. 8+207 pp. Cloth. M. 2.50

Grévy (A.). Géométrie élémentaire. Classes de 5e B et 3e B. 6e édition. Paris, Vuibert, 1914. 16mo. 396 pp. Fr. 3.00

Grignon (A.). Corrigé des exercices et problèmes d'algèbre. Paris, Gigord, 1914. 16mo. 304 pp.

Lazzeri (G.). Teoria elementare delle sezioni coniche. Livorno, Giusti, 1914. 16mo. 7+120 pp. L. 1.00

Manuel (G.). Deux cents problèmes et questions de théorie des examens du brevet élémentaire. 1re série. 7e édition. Paris, Hachette, 1914. 16mo. 135 pp. Fr. 1.00

MARSH (H. W.). Technical trigonometry. Being Volume IV of Constructive Text-Book of Practical Mathematics. New York, Wiley, 1914. 8vo. 10+232 pp. Cloth. $1.50

PASTOUR (J.). Leçons progressives de géométrie élémentaire. Paris, Belin, 1914. 12mo. 271 pp. Fr. 2.25

PERROTT (A. D.). See BORCHARDT (J.).

RÉBOUIS (A. G.). See CAMMAN (P.).

REPERTORIO di matematiche e fisica elementari, per G. L. 8a edizione. Livorno, Giusti, 1914. 24mo. 156 pp. L. 1.00

SCORZA (G.). Complementi di geometria. Volume I. (Collezione scolastica Laterza III.) Bari, Laterza, 1914. 8vo. 8+241 pp. L. 3.00

TESTI (G. M.). Elementi di matematica. Fascicolo VI. 5a edizione riveduta. Livorno, Giusti, 1914. 16mo. 112 pp. L. 0.90

WAELE (C. DE). See DALLE (A.).

WALTHER (F.). Mathematischer Lehr- und Uebungsgang für Lehrerbildungsanstalten. Leipzig, Brandstetter, 1914. 8vo. 58 pp. M. 2.00

III. APPLIED MATHEMATICS.

ALLEN (C. F.). Field and office tables; railroad curves and earthwork. 5th edition, revised. Two volumes in one volume. New York, McGraw-Hill, 1914. 516 pp. $3.00

ANDERSON (W. B.). Mechanics and heat. New York, McGraw-Hill, 1914. 8vo. 349 pp. Cloth. $2.00

ANNUAIRE pour l'an 1914, publié par le Bureau des Longitudes, avec des notices scientifiques de MM. Hatt, Bigourdan et Baillaud. Paris, Gauthier-Villars, 1914. 16mo. 700 pp. Fr. 1.50

ANTILLI (A.). Disegno geometrico. 4a edizione aumentata. Milano, Hoepli, 1914. 24mo. 11+84 pp. L. 2.00

AUERBACH (F.). Die graphische Darstellung. Eine allgemeinverstandliche, durch zahlreiche Beispiele aus allen Gebieten der Wissenschaft und Praxis erlauterte Einfuhrung in den Sinn und den Gebrauch der Methode. (Sammlung "Aus Natur und Geisteswelt," Nr. 437.) Leipzig, Teubner, 1914. 12mo. 6+97 pp. Cloth. M. 1.25

BAROLIN (J. C.). Der Hundertstundentag. Vorschlag zu einer Zeitreform unter Zugrundlegung des Dezimalsystems. Mit einem Geleitwort von I. Hugershoff. Wien, Braumuller, 1914. 8vo. 144 pp. M. 1.50

BARTHEL (E.). Vertikaldimension und Weltraum. Neue Beweise gegen die Kugelgestalt der Erde. Leipzig, Hillmann, 1914. 8vo. 28 pp. M. 0.75

BATES (E. L.) and CHARLESWORTH (F.). Mechanics for builders. Part II. London, Longmans, 1914. 8vo. 8+241 pp. 4s.

BÉCOURT (L.) et SAY (J. B.). Le dessin technique. Série C, Cahier I: Lavis théorique. 5e édition. Série D: Perspective. Cahier I. 7e édition. Série F: Mécanique. Cahiers VII et VIII. Paris, Hachette, 1914. 16mo. Chaque cahier, Fr. 1.00

BIRKELAND (B. J.) und HESSELBERG (T.). Der Widerstand einer bewegten Kugel im homogenen Medium. (*Videnskapsselskapets skrifter.*) Kristiania, Dybwad, 1914. 8vo. 15 pp. M. 0.50

Bryan (G. H.). Die Stabilität der Flugzeuge. Aus dem Englischen ubersetzt von H. G. Bader. Berlin, Springer, 1914. Cloth. M. 7.00

Charlesworth (F.). See Bates (E. L.).

Dähne (A.). Bausteine zur Flugbahn- und Kreiseltheorie. Berlin, Eisenschmidt, 1914. M. 1.50

Eucken (A.). Die Theorie der Strahlung und der Quanten. Verhandlungen auf einer von E. Solvay einberufenen Zusammenkunft. Halle a. S., Knapp, 1914. 12+405 pp. M. 15.60

Fantasia (P.). Formule ed esercizi per l'applicazione del metodo di minimi quadrati alla topografia. Milano, Hoepli, 1914. 24mo. 16+339 pp. L. 4.00

Föppl (L.). Rotierendes Ei auf horizontaler Unterlage. (Habilitationsschrift.) Wurzburg, 1914.

Forchheimer (P.). Hydraulik. Leipzig, Teubner, 1914. 8vo. 10+566 pp. Cloth. M. 19.00

Gans (R.). Einfuhrung in die Vektoranalysis. Mit Anwendungen auf die mathematische Physik. 3te Auflage. Leipzig, Teubner, 1914. 8vo. M. 3.40

Geitel (M.). Schopfungen der Ingenieurtechnik der Neuzeit. (Aus Natur und Geisteswelt. Nr. 28.) Leipzig, Teubner, 1914, 8vo. M. 1.25

Gouard (E.) et Hiernaux (G.). Cours élémentaire de mécanique industrielle. Tome I. 2e édition, corrigée et augmentée. Paris, Dunod et Pinat, 1914. 16mo. 8+386 pp. Cartonné. Fr. 4.50

Guldner (H.). Das Entwerfen und Berechnen von Verbrennungskraftmaschinen und Kraftgasanlagen. 3te, neubearbeitete Auflage. Berlin, Springer, 1914. Cloth. M. 32.00

Haussner (R.). Darstellende Geometrie. 1ter Teil: Elemente; ebenflachige Gebilde. 3te, vermehrte und verbesserte Auflage. (Sammlung Goschen. Neue Auflage. Nr. 142.) Berlin, Goschen, 1914. 8vo. 207 pp. Cloth. M. 0.90

Hawks (E.). Astronomy. London, Milner, 1914. 12mo. 120 pp. Cloth. 1s.

Hesselberg (T.). See Birkeland (B. J.).

Hiernaux (G.). See Gouard (E.).

Hoppe (M. L.). Die Abhängigkeit der Wirbeltheorie des Descartes von Wilhelm Gilberts Lehre vom Magnetismus. (Diss.) Halle, 1914.

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THE TWENTY-FIRST SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

By invitation of Brown University, the twenty-first summer meeting of the Society was held at that institution on Tuesday and Wednesday, September 8–9, 1914, in connection with the celebration of the one hundred and fiftieth anniversary of the founding of the university. Two sessions were held on Tuesday and a morning session on Wednesday, the following fifty-two members being in attendance:

Professor R. C. Archibald, Dr. Ida Barney, Professor G. D. Birkhoff, Professor J. W. Bradshaw, Professor B. H. Camp, Dr. A. B. Chace, Dr. A. Cohen, Professor F. N. Cole, Dr. C. F. Craig, Mr. C. H. Currier, Mrs. E. B. Davis, Professor E. W. Davis, Professor S. C. Davisson, Professor W. P. Durfee, Professor L. P. Eisenhart, Mr. H. J. Ettlinger, Professor G. C. Evans, Professor H. B. Fine, Professor T. S. Fiske, Dr. W. C. Graustein, Professor C. N. Haskins, Professor E. R. Hedrick, Professor E. V. Huntington, Professor W. A. Hurwitz, Dr. Dunham Jackson, Dr. R. A. Johnson, Professor Edward Kasner, Professor C. J. Keyser, Mr. John McDonnell, Dr. J. K. Lamond, Professor H. P. Manning, Professor G. A. Miller, Professor G. D. Olds, Dr. F. W. Owens, Professor Alexander Pell, Professor Anna J. Pell, Professor A. D. Pitcher, Professor T. M. Putnam, Professor R. G. D. Richardson, Professor E. D. Roe, Jr., Mrs. E. D. Roe, Jr., Professor D. A. Rothrock, Dr. H. M. Sheffer, Professor H. E. Slaught, Professor E. R. Smith, Professor J. N. Van der Vries, Professor E. B. Van Vleck, Professor A. G. Webster, Professor J. H. M. Wedderburn, Professor H. S. White, Professor Ruth G. Wood, Professor J. W. Young.

The President of the Society, Professor E. B. Van Vleck, occupied the chair at the morning sessions; Vice-President L. P. Eisenhart presided at the Tuesday afternoon session. The Council announced the election of the following persons to membership in the Society: Mr. Lincoln K. Adkins, University of Minnesota; Dr. Lennie P. Copeland, Wellesley College; Mr. John W. Cromwell, Jr., Washington, D. C., High Schools; Professor Tsuruichi Hayashi, Tôhoku Imperial University, Sendai, Japan; Professor Claude I. Palmer, Armour

Institute of Technology; Mr. George A. Pfeiffer, Columbia University; Mr. Paul R. Rider, Yale University; Dr. Alfred Rosenblatt, University of Cracow; Miss Caroline E. Seely, Columbia University. Eleven applications for membership in the Society were received.

In order that those wishing to do so may attend the winter meeting of the Chicago Section, the meeting of Section A of the American Association, and the annual meeting of the Society (at which President Van Vleck will deliver his presidential address), it was voted to suspend By-Law III to the extent of holding the annual meeting not later than January 5, 1915. A committee, consisting of Professors M. W. Haskell, H. F. Blichfeldt, L. E. Dickson, O. D. Kellogg, and the Secretary, was appointed to arrange for holding the summer meeting at San Francisco in 1915.

It was decided to issue only a List of Officers and Members next year, instead of the complete Annual Register.

The first session, on Tuesday morning, opened with an address of welcome by Chancellor A. B. Chace of Brown University. The social programme included a tea in the John Carter Brown Library on Tuesday afternoon and luncheon in Rockefeller Hall on Wednesday, both given by Professor N. F. Davis. The University gave a dinner to the Society at the University Club on Tuesday evening, which was concluded by an address by President Faunce and an interesting account by Professor Carl Barus of the "Historical development of the modern theories of physics." A vote of thanks was tendered to the University and its officers for their generous hospitality. Wednesday afternoon was devoted to an excursion to Newport.

The following papers were presented at this meeting:

(1) Dr. F. M. MORGAN: "A plane cubic Cremona transformation and its inverse."

(2) Professor L. P. EISENHART: "Conjugate systems with equal tangential invariants and the transformation of Moutard."

(3) Professor C. E. LOVE: "Singular integral equations of the Volterra type."

(4) Professor O. E. GLENN: "Modular invariant processes."

(5) Professor L. E. DICKSON: "Invariants, seminvariants and covariants of the ternary and quaternary quadratic form modulo 2."

(6) Professor L. E. DICKSON: "The points of inflexion of a plane cubic curve."

(7) Professor L. E. DICKSON: "A fundamental set of modular invariants of the system of the binary cubic, quadratic, and linear form."

(8) Professor L. E. DICKSON: "Invariants in the theory of numbers."

(9) Professor F. B. WILEY: "Proof of the finiteness of the modular covariants of a system of binary forms and cogredient points."

(10) Professor E. V. HUNTINGTON: "The theorem of rotation in elementary dynamics."

(11) Dr. R. D. BEETLE: "Congruences associated with a one-parameter family of curves."

(12) Professor G. C. EVANS: "The non-homogeneous parabolic differential equation."

(13) Dr. R. A. JOHNSON: "The conic as a space element."

(14) Professor W. A. HURWITZ and Dr. L. L. SILVERMAN: "On the consistency and equivalence of certain definitions of summability."

(15) Professor MAXIME BÔCHER: "The method of successive approximations for linear differential systems."

(16) Professor MAXIME BÔCHER: "The smallest characteristic numbers in a certain exceptional case."

(17) Professor B. H. CAMP: "On the series obtained by term-wise integration."

(18) Professor G. A. MILLER: "On the φ-subgroup of a group."

(19) Dr. T. E. MASON: "On functions transcendentally transcendental with respect to a given realm of rationality."

(20) Dr. T. E. MASON: "Mechanical device for testing Mersenne numbers for primes."

(21) Mr. H. S. VANDIVER: "On Bernoulli's numbers, Fermat's quotient and last theorem."

(22) Professor L. C. KARPINSKI: "An early algorism."

(23) Professor H. S. WHITE: "Triple-systems on 31 letters; a reconnoissance."

(24) Miss L. D. CUMMINGS: "The trains for 42 non-congruent triple-systems on 15 elements."

(25) Professor J. H. M. WEDDERBURN: "On matrices whose coefficients are entire functions."

(26) Professor E. R. SMITH: "A problem in the fitting of polynomial curves to certain kinds of data."

(27) Dr. H. R. KINGSTON: "Metric properties of nets of plane curves."

(28) Professor G. D. BIRKHOFF: "The iterated transformation of a plane into itself."

(29) Professor W. B. FITE: "Prime power groups in which every commutator of prime order is invariant."

(30) Professor EDWARD KASNER: "Transversality for double integrals in the calculus of variations and for contact transformations."

(31) Professor EDWARD KASNER: "The decomposition of conformal transformations into factors of period two."

(32) Professor R. G. D. RICHARDSON: "A new boundary value problem for linear hyperbolic differential equations of the second order."

(33) Dr. JOSEPH ROSENBAUM: "Mixed linear integral equations over a two-dimensional region."

(34) Professor D. C. GILLESPIE: "Cauchy's definition of a definite integral."

Dr. Kingston's paper was communicated to the Society by Professor Wilczynski. Dr. Rosenbaum was introduced by Professor Hurwitz. In the absence of the authors, Miss Cummings's paper was read by Professor White and the papers of Dr. Morgan, Professor Love, Professor Glenn, Professor Dickson, Professor Wiley, Dr. Beetle, Professor Bôcher, Dr. Mason, Mr. Vandiver, Professor Karpinski, Dr. Kingston, Professor Fite, and Professor Gillespie were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper Dr. Morgan studies the properties of the plane cubic Cremona transformation $S = (\frac{x}{y_1(x_1^2+y_1^2)} \frac{y}{x_1(x_1^2+y_1^2)} \frac{z}{x_1y_1z_1})$ and its inverse $T = (\frac{x_1}{xy^2} \frac{y_1}{yx^2} \frac{z_1}{z(x^2+y^2)})$. Both of these transformations are examples of an α_i coinciding with an α_k. When written in non-homogeneous coordinates these transformations present a simple ruler and compass construction by means of which one may construct a cissoid, a four-leaved rose, a lemniscate, and other higher plane curves.

2. When the tangential coordinates of a surface S are solutions of an equation of the form

$$(1) \qquad \frac{\partial^2\theta}{\partial u \partial v} + \frac{\partial \log \rho}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log \rho}{\partial u}\frac{\partial \theta}{\partial v} = 0$$

the parametric coordinate system is said to have equal tangential invariants. Professor Eisenhart shows that there exist transformations of S into surfaces S_1 such that the parametric system on S_1 is conjugate with equal tangential invariants, and moreover the lines joining corresponding points on S and S_1 form a congruence whose developables meet these surfaces in the parametric curves. In this case the surfaces S and S_1 are said to be in the relation of a transformation Ω. The surfaces associate to S and S_1 with the asymptotic lines of the former corresponding to the parametric conjugate systems on the latter can be so placed in space that they are the focal surfaces of a W-congruence. If S and S_1 are in the relation of a transformation Ω, the tangent planes to these surfaces at corresponding points are harmonic to the focal planes of the congruence of lines of intersection of these planes. Of particular interest is the case in which the latter congruence is normal. To this class belong the transformations of surfaces of Voss discussed by the author in the July number of the *Transactions*. The transformations of surfaces with isothermal spherical representation of their lines of curvature established in the ninth volume of the *Transactions* are transformations Ω and the only ones such that S and S_1 are the two sheets of the envelope of a two-parameter family of spheres. If S_1 and S_2 are two transforms of S, there exist an infinity of surfaces S' each of which is in the relation of a transformation Ω with S_1 and S_2. Moreover, if M, M_1, M_2 are the corresponding points of S, S_1, S_2, the corresponding points of the surfaces S' lie on a conic in the plane of M, M_1, M_2; and the tangent planes to the surfaces S' pass through the point of intersection of the corresponding tangent planes to S, S_1, S_2.

The transformation of Moutard of equations of the type (1) plays an important rôle in the transformations Ω, as also in the transformations K of conjugate systems with equal point invariants announced by the author at the April meeting of the Society. The paper will appear in the *Palermo Rendiconti*.

3. Professor Love's paper proves the existence of a continuous solution $\varphi(x)$ of the singular Volterra equation of the second kind

$$\varphi(x) = f(x) + \int_x^{\infty} K(x, t)\varphi(t)dt$$

under various hypotheses concerning the functions $f(x)$ and $K(x, t)$. In each case the behavior of $\varphi(x)$ for large values of x is also considered. The paper will appear in the October number of the *Transactions*.

4. Under linear transformations on the variables x_i $(i=1, \cdots, m)$ whose coefficients are positive residues of a prime number p the functions $x_i^{p^t}$ $(i = 1, \cdots, m)$ are cogredient to the variables. Professor Glenn proves this and shows that it leads to the fact that the polar operator

$$x_1^{p^t}\frac{\partial}{\partial x_1} + \cdots + x_m^{p^t}\frac{\partial}{\partial x_m},$$

applied to any covariant of an algebraic m-ary form f, gives a formal modular covariant of f. Modular transvectants are also defined by means of such cogredient expressions, and their properties discussed. Application is made to the derivation of an irreducible system of formal modular concomitants of a linear form, consisting of two invariants and six covariants (cf. Dickson, Madison Colloquium Lectures, page 40).

Since the transformation on the x's induces a similar group of transformations on the coefficients $a_0, a_1, \cdots$, we have a dual theory of the modular Aronhold operators

$$a_0^{p^t}\frac{\partial}{\partial a_0} + \cdots + a_\mu^{p^t}\frac{\partial}{\partial a_\mu}.$$

5. The object of the first paper by Professor Dickson is to provide an improved method for finding the leading coefficient of a modular covariant of a form or forms in three or more variables. For the case of a ternary quadratic form modulo 2, the method employed in the Madison Colloquium Lectures, pages 77–79, is sufficiently brief, but is not as expeditious as the new method, especially in more difficult problems. The paper will appear in the BULLETIN.

6. In the second paper by Professor Dickson, an elementary and largely novel exposition is given of the classic theory of the inflexion points of a plane cubic curve without singular points. Then the Galois group of the equation X for the abscissas of the nine inflexion points is determined for special cubic curves as well as for the curve in whose equation the

coefficients are independent variables. The knowledge of the group and its properties is applied to a complete analysis of the solution by radicals of the equation X. The paper will appear in the December number of the *Annals of Mathematics.*

7. The third paper by Professor Dickson shows how simple and remarkably effective is the theory of modular invariants based upon the separation of equivalent sets of ground forms into classes (cf. Madison Colloquium Lectures), by treating the simultaneous modular invariants of the binary cubic, quadratic, and linear form. The paper will appear in the *Quarterly Journal of Mathematics.*

8. A. Hurwitz* introduced the concept of formal invariants of binary forms (whose coefficients are independent variables) under the group of all binary linear transformations with integral coefficients taken modulo p, a prime. Aside from the cases† of quadratic and cubic binary forms, and a theorem by Miss Sanderson‡ connecting the subject with modular invariants, no progress has been made towards a general theory. A remarkably simple and effective method of constructing formal invariants is given in the fourth paper by Professor Dickson. In the case of a single form f in n variables, the method consists in forming the symmetric functions of certain like powers of the values of f at the real points in modular space. The power used is such that f has the same value at $(x_1, \cdots, x_n)$ as for $(\rho x_1, \cdots, \rho x_n)$, where ρ is any integer. In the case of several forms f_i, we employ the symmetric functions of $\phi(v_1, v_2, \cdots)$, in which $v_1, v_2, \cdots$ range over the sets of values of powers of $f_1, f_2, \cdots$ at the real points, while ϕ is any polynomial with integral coefficients. We may also employ the points whose coordinates are in a Galois field of order p^k. From the special cases examined, it appears that the method furnishes a fundamental system of formal invariants. There is an immediate extension to the case of seminvariants and to the general case of invariants under any modular group. The paper will appear in the *Transactions.*

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* *Archiv der Mathematik und Physik* (3), vol. 5 (1903), p. 25.

† Dickson, The Madison Colloquium Lectures, pp. 41–54.

‡ *Transactions*, vol. 14 (1913), p. 490.

9. The question of the finiteness of the modular covariants of a system of forms and cogredient points arises naturally at this time in view of the fact that Professor Dickson has proved (*Transactions,* 1913) the finiteness of the modular covariants of a system of forms without cogredient points and Professor Krathwohl secured in his Chicago dissertation (*American Journal,* October, 1914) a fundamental set of modular covariants of two cogredient binary points. In the present paper Professor Wiley secures the theorem:

The set of all modular covariants of any system of forms in x_{01}, x_{02} and the cogredient points x_{k1}, x_{k2} $(k = 1, \cdots, n)$ is finite in the sense that they are all rational integral functions, with integral coefficients taken modulo p, of a finite number of covariants of the set.

This paper will appear in the *Transactions.*

10. In the theory of the plane motion of a rigid body, the equation $Fr = (W/g)k^2d\omega/dt$, where $d\omega/dt$ is the angular acceleration of the body at the instant considered, plays an important rôle. In this equation, Fr is the sum of the moments of the external forces and k the radius of gyration of the body about a suitably chosen point Q. The question at once arises, what is meant by a suitably chosen point in this case. All the books agree that the equation is true if Q is fixed in the body and fixed in space, or if Q coincides with the center of mass. Many books state that the equation is true also when Q is the instantaneous center of rotation, and some that it is true when Q is any point whatever. In view of these discrepancies in the current elementary textbooks, it seems worth while to review the facts concerning this important equation. In order to do this most effectively, Professor Huntington states and proves two distinct theorems, one for the case in which Q is fixed in the body, the second for the case in which Q is fixed in the plane.

I. If Q is a point fixed in the body, then the equation $Fr = (W/g)k^2d\omega/dt$ is true: (1) when Q is fixed in the plane, (2) when Q is moving with constant velocity in a straight line, (3) when Q is the center of mass, (4) when Q has an acceleration whose direction at the instant in question passes through the center of mass; and in no other cases. In particular, it is *not* in general legitimate to "take moments about the instantaneous center."

II. If Q is fixed in space, $Fr = (W/g)d(c\bar{v})/dt + (W/g)k_0^2 d\omega/dt$, where Fr is the sum of the moments of the external forces about the point Q at the instant in question, $c\bar{v}$ is the moment of the velocity of the center of mass about Q at that instant, and k_0 is the radius of gyration about the center of mass.

These two theorems are of course implicitly contained in any standard treatise on the subject; but they do not seem to be explicitly stated in this form in any of the elementary textbooks. The proofs are here given in a very direct and simple form.

11. With a one-parameter family of curves may be associated a number of rectilinear congruences. The congruence formed by the tangents to the curves has been very extensively studied, and there are many theorems concerning the relations between the properties of this congruence and those of the curves and of the surface on which they lie. The other congruences naturally associated with a one-parameter family of curves have received little attention. In this paper, Dr. Beetle considers some of these other congruences, giving particular attention to those formed by the principal normals and binormals of the curves.

In connection with certain properties of the congruences, it is of interest to consider those surfaces on which the asymptotic lines in one system, or both, are geodesic parallels. It is shown that the determination of all such surfaces requires the solution of a rather complicated partial differential equation of the fourth order. Several characteristic properties of these surfaces are found.

12. Professor Evans considers the non-homogeneous partial differential equation $\partial u/\partial t - \partial^2 u/\partial x^2 = f(x, t)$ and the function which is its principal solution, when the proper derivatives exist, with the object of generalizing the conditions on $f(x, t)$ necessary for the existence of these derivatives. According to the nature of the parabolic equation, such conditions should be asymmetrical in character. It is proved that those derivatives exist at the point (x_1, t_1) provided that $f(x, t)$ is finite and integrable in a suitably defined region, and at (x_1, t_1), is continuous and satisfies the condition $|f(x, t) - f(x_1, t)| \leqq N|x - x_1|^\tau$, where N and τ are positive numbers, not zero.

13. Dr. Johnson uses as coordinates of the conic in three-

dimensional space the elements of the discriminant of its tangential equation; in other words, a conic is represented by ten homogeneous coordinates which are elements of a vanishing four-rowed symmetrical determinant. Thus there is a one-to-one correspondence between the conics of space and points of a quartic hypersurface in space of nine dimensions. The equations, in terms of these coordinates, which correspond to various projective relations between conics and other figures, such as points, lines, planes, quadrics, are developed for numerous simple cases. After a brief consideration of the relation of apolarity between quadrics, a detailed discussion is given of the possible linear systems of conics.

14. In the present paper Professor Hurwitz and Dr. Silverman, on the ground of previous work by Silverman, Toeplitz, and Schur, establish a correspondence between certain functions $f(z)$ and certain definitions of summability of a divergent sequence. If $f(z)$ is analytic within and on the boundary of the circle C of radius $\frac{1}{2}$ about the point $\frac{1}{2}$, the corresponding definition is a generalization of convergence, i. e., correctly evaluates any convergent sequence. All such definitions are consistent, i. e., if two of them evaluate a certain sequence, they give it the same value. Two definitions are equivalent, i. e., evaluate exactly the same sequences, provided the corresponding functions have the same zeros with the same multiplicities in C. Further properties of the definitions in question are derived; the equivalence of the Cesàro and Hölder definitions of the same order appears as an incidental result. The question of equivalence of different definitions has previously received some slight attention; it is believed that the question of consistency is here studied for the first time.

15. The results partially announced by Professor Bôcher at the Cambridge International Congress in 1912 concerning the scope of the method of successive approximations when applied in a very general manner to ordinary linear differential systems are here developed in detail.

16. Professor Bôcher's second paper appeared in full in the October BULLETIN.

17. It has long been known that, if a series is uniformly convergent, the series obtained by integrating it term-wise

is also uniformly convergent. In recent years more general sufficient conditions for term-wise integrability have been discovered, and finally, by Vitali and W. H. Young, conditions which are both necessary and sufficient. If a series which is not uniformly convergent be integrable term-wise over a variable interval, the new series may not be uniformly convergent, but it has certain other characteristics which may be classified according to which of these sufficient conditions the original series satisfies. Furthermore, successive integrations over the same interval lead to progressively simpler series.

In Professor Camp's paper several general theorems dealing with this subject are developed. A particular result, capable of being stated concisely, is that, if a series may be integrated term-wise in the variable interval (a, x), then the new series may be integrated term-wise, even after being multiplied through by any function of x which is absolutely integrable in Lebesgue's sense.

18. All the operators of any group G can be divided into two categories having no common operator by putting into one category (A) all those operators of G which can occur in one of the possible sets of independent generators and into the other category (B) those which do not have this property. The category (B) is a subgroup of G, and this subgroup was called by G. Frattini the φ-subgroup of G. It is always the direct product of its Sylow subgroups and hence it is always solvable. The φ-subgroup of G may also be defined as the cross-cut of all the possible maximal subgroups of G. A simple example of a φ-subgroup is the subgroup of order p^{m-1} in the cyclic group of order p^m, p being any prime number.

Professor Miller developed a number of theorems relating to the φ-subgroups. In particular, he proved that the order of the φ-subgroup of any abelian group of order p^m is $p^{m-\alpha}$, whenever p^α is the order of the subgroup generated by all of the operators of order p. A necessary and sufficient condition that the φ-subgroup of an abelian group be identity is that all the Sylow subgroups of this abelian group are of type $(1, 1, 1, \cdots)$. If a group can be represented as a primitive substitution group its φ-subgroup is identity; in particular, the φ-subgroup of every simple group of composite order is identity.

19. Dr. Mason finds conditions that first and second order linear difference and q-difference equations and Riccati difference and q-difference equations have only solutions that are transcendentally transcendental with respect to a general realm of rationality which includes the coefficients. These conditions are expressed in the possibility of certain auxiliary equations having solutions algebraic in the realm of rationality considered. The results for these equations in the realm of rationality x are to be found in papers by Tietze, Stridsberg, and a previous paper by Dr. Mason.

20. Lucas has given a method of testing Mersenne numbers for primes, in which he uses multiplication in the binary system of numeration. Dr. Mason shows how multiplication in this system can be reduced to counting correspondences, and suggests a mechanical device to do the counting and recording.

21. Mr. Vandiver obtains some general theorems respecting Fermat's quotient and Bernoulli's numbers and in particular the following relation:

$$\frac{(m^a - 1)h_a}{a} \equiv \sum_{n=1}^{p-1} y_n n^{a-1} \pmod{p},$$

where $1 < a < p - 1$, $1 < m < p$ (p a prime),

$$y_n \equiv -\frac{n}{p} \pmod{m}, \quad m > y_n \geqq 0,$$

$$h_{2a} = (-1)^{a-1} B_a, \quad h_{2a+1} = 0,$$

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad \text{etc.}$$

By means of this theorem it is then shown that Kummer's criteria in reference to Fermat's last theorem may be replaced by the following conditions:

$$\sum_{n=1}^{p-1} y_n n^{p-2} t^n \equiv 0 \pmod{p} \quad (m = 2, 3, \cdots, p-1).$$

(See notation used in *Crelle*, volume 144, page 314.) A number of other transformations of the Kummer congruences are also given.

22. The Latin treatises on the Hindu art of reckoning which are earlier than the fourteenth century are sufficiently rare so that the addition even of anonymous works to the available literature is desirable. An algorism which bears internal evidence that it was one of the very early treatises on this subject has been found by Professor Karpinski in the British Museum. Royal MS .15 B IX contains the algorism in question; this manuscript is dated by the experts of the museum as of the end of the twelfth or beginning of the thirteenth century. A somewhat similar work, directly related to this algorism, is found in the Egerton MS 2261, fol. 225^b–227^b, and a second copy in Codex Paris, 10252, fol. 68^a–70^b. The copy in Paris, written 1476, was found by Björnbo and is included in his list of medieval mathematical works which is preserved in Stockholm. A study is made of all three of the manuscripts.

This paper will appear in the *Bibliotheca Mathematica.*

23. The number of different triad systems on n letters is known only for the small values of n. Reiss proved that at least one system exists for every value of $n = 6t + 1$ or $n = 6t + 3$. E. H. Moore proved that at least two incongruent systems exist for every such n above 13. Zulauf showed two distinct systems for 13. Miss Cummings has distinguished 24 systems for $n = 15$ without exhausting the possibilities. In the present paper, from one simple theorem and certain data not yet published, Professor White demonstrates the existence of a very large number, above 10^{13}, of systems of a particular kind on 31 letters.

24. Miss Cummings has applied to triple-systems on 15 elements the method for determining the non-congruency of triple-systems given by Professor H. S. White in the *Transactions* of January, 1913. The paper shows the 203 distinct types of trains which occur in 42 non-congruent systems Δ_{15}.

25. In this paper, Professor Wedderburn extends the methods ordinarily used in reducing to its normal form a matrix whose coefficients are polynomials in a single variable so as to make these methods applicable when polynomials are replaced by entire functions or, more generally, by functions holomorphic in a simply connected region.

26. If to a series of values of x which are in an arithmetical progression there corresponds a series of observed values of y, and if it is desired to determine by the method of least squares or by the method of moments the coefficients of the polynomial

$$y = a_0 + a_1x + a_2x^2 + \dots + a_kx^k$$

which give the best relation between x and y, the problem can be simplified by evaluating the determinant

$$D_k(n) = \begin{vmatrix} \sum_0^n x^0 & \sum_0^n x & \sum_0^n x^2 & \dots \sum_0^n x^k \\ \sum_0^n x & \sum_0^n x^2 & \sum_0^n x^3 & \dots \sum_0^n x^{k+1} \\ \dots & \dots & \dots & \dots \\ \sum_0^n x^k & \sum_0^n x^{k+1} & \sum_0^n x^{k+2} & \dots \sum_0^n x^{2k} \end{vmatrix}.$$

By separating the determinant into two factors Professor Smith finds the roots of $D_k(n)$ and consequently its factors. It follows then that when n is less than $k + 1$, $D_k(n)$ vanishes, when $n = k + 1$, $D_k(n)$ is the square of a simple Vandermondian, and finally when n is greater than $k + 1$,

$$D_k(n) = C_kn^k \prod_{v=0}^{v=k-1} (n - k + v)^{v+1}(n + k - v)^{v+1},$$

where C_k is a constant for any value of k. Having found this value for the determinant, simple expressions are obtained for the coefficients $a_0, a_1, a_2, \dots, a_k$ which depend only on the observed values of y and on the number of the observations.

27. In a memoir entitled "One-parameter families and nets of plane curves,"* Wilczynski has discussed the projective differential properties of nets of plane curves, by means of a completely integrable system of three partial differential equations of the second order. In Dr. Kingston's paper the foundation is laid for a study of the metric differential properties of such nets. In order to accomplish this, it becomes necessary to consider, besides the coefficients of the partial differential equations used by Wilczynski, the coefficients E, F, and G of the square of the element of arc when this is ex-

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* *Transactions*, vol. 12 (1911), pp. 473–510.

pressed in terms of the parameters of the net curves as coordinates. All of these quantities taken together, which must moreover satisfy certain relations, determine a net uniquely except for a motion combined with a reflection. The application of the general theory to the particular cases of orthogonal and isothermal nets gives rise to interesting results.

28. The iterated transformation of a plane into itself leads to hitherto untreated problems in point set theory which stand in vital relation to certain dynamical questions. The problem considered by Professor Birkhoff is that of characterizing as far as possible the sequence of positions taken by an arbitrary point of the plane under repetition of the transformation and its inverse. Some of the results obtained run parallel to those given by him in an earlier paper.*

29. Professor Fite's paper concerns itself with those prime power groups in which every commutator of prime order is invariant. It is shown that if G is of order p^m (p an odd prime) and any cogredient has a cyclic central, the second central must contain a commutator of order p that is not invariant in G. In fact, if any cogredient contains a non-invariant operation of order p, the second central must contain a commutator of order p that is not invariant in G. Of various other results obtained the two following may be mentioned: (1) If every operation of the second central whose order does not exceed p^r is invariant in G, every operation of G whose order does not exceed p^r is invariant. (2) If the commutator subgroup is cyclic and the $(k-1)$th central is of class k_1, G must be of class k, where $k \leqq 3k_1$.

30. In connection with the minimizing of double integrals, Kneser's theory of transversality gives rise to a correspondence between the surface elements and the lineal elements at any point of space. This correspondence is not entirely arbitrary in character. Professor Kasner obtains a simple geometric property which furnishes a criterion as to when a given correspondence can be regarded as a transversality correspondence. In particular the only linear correspondences satisfying the test are polarities.

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* *Bulletin Soc. Math. de France*, vol. 40 (1912), pp. 305–323.

The general result obtained is also applied to the theory of infinitesimal contact transformations, and indirectly to the transversality correspondences associated with simple integrals in space. Certain connections between simple and double integrals are thus derived.

31. The conformal transformations discussed in Professor Kasner's second paper are assumed to be regular in the neighborhood of the origin. They are thus expressed by power series in $x + iy$ or $x - iy$ without absolute term. It is shown that there are two types of transformations of period two: the first, in $x + iy$, are called conformal involutions; and the second, in $x - iy$, are identified with symmetries (in the general sense of Schwarz). The main results of the paper are criteria as to when a given conformal transformation can be regarded as the product of either two involutions, or two symmetries. The classes of transformations thus obtained are very large, only a finite number of restrictions being imposed on the coefficients of the corresponding power series. As in the author's earlier results in conformal invariants, distinctions arise according to the rationality or irrationality of certain angles determined by the coefficients.

32. For linear hyperbolic differential equations of the second order in two variables, the boundary value problem ordinarily considered is that of finding a solution $u(x, y)$ which on a piece of a curve has prescribed values for u and $\partial u/\partial n$. The simplest case of the new problem studied consists in determining whether there exist parameter values λ of the equation

$$(1) \qquad \begin{gathered} (pu_x)_x - (qu_y)_y + \lambda u = 0 \\ [p(x, y) \geqq k_1 > 0, \quad q(x, y) \geqq k_2 > 0] \end{gathered}$$

such that the corresponding solutions vanish on the boundary of a region $a_1 \leqq x \leqq a_2$, $b_1 \leqq y \leqq b_2$. Closely associated with this problem are new problems in linear equations and in quadratic forms each with an infinite number of variables, as well as with n variables where n is allowed to increase indefinitely. Professor Richardson has shown that there is a denumerable infinity of parameter values λ of the equation (1) for which there are solutions of the type sought. This set of values has limiting points at both positive and negative infinity as well as at one or more finite points. Under restrictions similar to those usually imposed, a function $f(x, y)$ may be developed in a series of solutions with Fourier coefficients.

33. Following out methods previously used by Professor W. A. Hurwitz in discussing mixed linear integral equations in one dimension, Dr. Rosenbaum in this paper establishes similar results for the case of two dimensions. The unknown function appears under integral signs operating over a plane region and over curves, and the values of the unknown function at special points also appear in the equation. The adjoint system of equations now involves as unknowns one function of two variables, several functions of one variable, and several constants. The notions of resolvent system, orthogonalization of principal solutions, and pseudo-resolvent system, receive similar generalizations.

34. The Riemann definition of the definite integral of a bounded function involves the values of the function at arbitrary points of the sub-intervals of a scheme of subdivision, while the Cauchy definition involves the values only at ends of sub-intervals. It is obvious that if the Riemann integral exists the Cauchy integral will exist; it is not immediately evident whether the converse is true or false. Professor Gillespie proves in this note that the two definitions are equivalent.

F. N. COLE,
Secretary.

INFINITE REGIONS IN GEOMETRY.

BY PROFESSOR EDWIN BIDWELL WILSON.

(Read before the American Mathematical Society, February 28, 1914.)

THE recent contribution by Professor Bôcher on "The infinite regions of various geometries"* puts me in mind of some ideas which I have long held on this subject and which I desire to offer to readers of his article.

The points which he most desires to make are:

1°, that when we are dealing with other geometries than the projective we should replace the infinite line of plane projective geometry and the infinite plane of projective geometry in three dimensions by such other infinite region as may be most appropriate to the geometry we are considering, and

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* This BULLETIN, vol. 20, pp. 185–200, January, 1914.

2°, that in particular when we introduce transformations which throw points to infinity we have not put our work in satisfactory form until we have made clear what infinite region we assume.

With the second of these points I am entirely in accord, and with the first, also, if it is properly understood; but I am inclined to think that Professor Bôcher has not put the matter of infinity in geometry in the best way* and that a literal adherence to his position would be unfortunate.

Just as he protests against the use of infinity in a carefree manner in different geometries, I wish on my part to protest against its careless use in any particular geometry. According to the view which I believe to be in fullest consonance with modern scientific attitudes toward geometry, there are no infinite (or ideal) points in the projective plane, no line at infinity in projective geometry, no infinite point in real inversive geometry, nor any pair of lines at infinity in complex inversive geometry.

To substantiate this point of view and to show at the same time the relation of my view to Professor Bôcher's I must give some definitions, in particular some definition of geometry. And as my predecessor gave no definition of what he meant by a geometry (except by implication), I shall begin by formulating a definition which seems to me to express his point of view.

Definition 1.—A geometry is the ensemble of those properties of configurations in the euclidean plane (and in its ideal extensions) which are invariant under the transformations of an r-parametered group†

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* There can be no doubt that Professor Bôcher is familiar with the views which I shall expound and that he had reasons which seemed best to him for sticking close to the exposition he chose. There is still a different presentation, more elementary than either of ours, which I have heard Professor F. S. Woods offer at a semi-public gathering and which I hope he may offer to the BULLETIN; for in a subject like infinity in geometry I believe that the maximum satisfaction comes only with a multiplicity of views.

† It is only for ease that we restrict the definition to the plane and omit from the statement such groups as that of the Bewegungen und Umlegungen (motions and orthogonal reflections).

$$X' = X'(X, Y; c_1, c_2, \cdots, c_r), \quad Y' = Y'(X, Y; c_1, c_2, \cdots, c_r).$$

Definition 2.—For any particular choice of the parameters such points (X, Y) as render X' or Y' or both infinite are said to be thrown to infinity, and the infinite region of the geometry is of the type of the locus of such points (X, Y) for general values of the parameters.

Except for the unusual precision relative to infinity and a phraseology introduced by Lie, this definition is old. Applied to the characterization of projective geometry it represents the point of view early in the science of projective geometry—Poncelet, Steiner, Chasles, Salmon, Fiedler, and Cayley might reasonably be associated with it. It is still of vital pedagogic importance when we desire to lead the student for the first time from his known euclidean geometry to the new projective. For the recapitulation theory seems here to apply; the natural entogeny is an accelerated phylogeny.

Beginning, however, with von Staudt and continuing through a long succession which we may at present terminate with Veblen and Young,* there has been a tendency to place projective geometry on its own feet, to define it in terms not extraneous to itself. Although such a development has not yet reached many other geometries we are, I think, sufficiently advanced in our point of view to regard the proper definition of a geometry as something like this:†

Definition 3.—Given a system containing a set of undefined symbols (one or more classes of elements, one or more relations) and a set of primitive propositions connecting them; the geometry of this system is constituted of the body of propositions logically deduced from the primitive propositions.

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* "A set of assumptions for projective geometry," *Amer. Journ. of Mathematics*, vol. 30, pp. 347–380 (1908)

† See E. V. Huntington's definition of abstract geometry on page 526 of his article "A set of postulates for abstract geometry, expressed in terms of the simple relation of inclusion," *Math. Annalen*, vol. 73, pp. 522–559 (1913). He restricts his definition to cover only the particular geometry he is expounding, but states that such a definition is applicable in other geometries also. It is entirely possible that the definition which we formulate should not be applied in general to a system as yet unformed, but should be applied anew in each particular instance after the system has been formed and then only if the person who forms the system desires to call it a geometry; for in no other way does it seem possible to include all systems which have been or may be called geometries without also including pretty much every deductive system. For restrictions see definitions by B. Russell, Principles of Mathematics, p. 372, and A. N. Whitehead, The Axioms of Projective Geometry, p. 5.

(If we desire to define a geometry as distinguished from a class of geometries, we should insist that the system be categorical.)

Definition 4.—Any class of elements selected from that class, or from one of those classes,* which enters into the definition of a geometry may be called a region of the geometry. The infinite region of the geometry would be, if it existed, a certain special region. The specialization would have to be effected by means of some property which belonged to the geometry and to which the concepts finite and infinite (in some of their many senses) were alternatively applicable.†

Why, upon the basis of Definitions 3 and 4, has the projective plane or projective geometry no infinite region? Simply because if we take any purely projective definition of projective geometry, there are no special regions whatsoever which are singled out from the rest of the plane; a fortiori there can be no infinite region. One of the things which we should be most careful to impress upon the student of projective geometry is that the projective plane is entirely homogeneous.

For a similar reason real euclidean geometry has no infinite region; it has no special points or lines in it. We may refer to Huntington's paper previously cited. The non-euclidean geometry developed in detail by Lewis and myself for the representation of the principle of relativity is also without an infinite region.‡

As to inversive geometry it may well be that as yet there is no exact formulation in postulates, but the possibility of such formulation is so evident that it is safe to say that inversive geometry has no infinite element or infinite region; the inversive plane is homogeneous. We might as well maintain that the surface of a sphere or spherical geometry, when defined by a system of postulates appropriate to the geometry,§

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* We could define mixed regions by collecting elements from different classes.

† We formulate no precise definition of infinite elements or infinite region because in most of the geometries which have been handled in the modern logical manner there is no infinite region

‡ Wilson and Lewis, "The space-time manifold of relativity," *Proceedings of the American Academy*, vol. 48, pp. 389–507. There are singular loci in the geometry, but no special points. If we consider as fundamental the right line and angle, we could regard the singular lines as infinite elements; but the advantage of such procedure is problematical.

§ For a definition which is not proved to be either complete or categorical see E. B. Wilson, "Seven lectures on spherical geometry," *Amer. Math. Monthly*, serially in 1904. In those lectures I made use of the idea of a possessed a special element (such as a north pole) as that the inversive plane had a special element such as the infinite point.*

As now, speaking from the geometric and logical viewpoints, we have abolished the infinite regions in some, and could abolish them in all, of the geometries of which Professor Bôcher speaks, it is necessary to answer the question: What and where are the infinite regions of which he speaks? They are ideal, they are perhaps nowhere. They arise algebraically through the becoming infinite of some function; this is accidental to our choice of coordinates.† They originate geometrically from the breaking down of a correspondence between the planes of two different geometries;‡ this latter is the really essential geometric fact.

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group of transformations in formulating the axioms, just as Lewis and I later adopted that point of view in our geometry of relativity, loc. cit. In view of the importance of the group concept in modern geometry it might be desirable that some of our eminent specialists in postulates should construct systems in which that concept was emphasized. It is unfortunate in some respects that groups have been tied so closely to analytical representations as far as their geometric uses go.

* Professor Bôcher in a footnote calls attention to the fact that in Study's long treatment of Das Apollonische Problem, *Math. Annalen*, vol. 49, pp. 497–542, there is not a word said concerning the nature of the infinite region. According to our reading of Study's work the reason that no mention of the infinite region occurs may well be that for him, as for us, there is no such region in the geometry of inversion. We may be reading our own ideas into Study's text (which would be a heinous offence on our part toward so illustrious a geometer), but we believe that he makes his point of view quite clear. He does not set up a categorical system of postulates for inversive geometry, but he does point out very precisely that one of the chief differences between his geometry and Mascheroni's is that the latter uses the center of a circle whereas he makes no use of it. As one of Study's fundamental constructions is to find the inverse of any point with respect to any circle, the center could be found if the point at infinity were in the system, and then there would be but little gain in banishing the use of the center. Furthermore Study makes very strong the point of view which actuates him in this article, as in other of his work, namely, the desire to remain completely within his geometry.

† It is interesting to observe that we cannot represent the points on a simple closed curve by a continuously varying parameter so that the correspondence shall be one-to-one. What we do is to use a parameter t subject to $t_0 \leqq t \leqq t_1$, which assigns to the same point the values t_0 and t_1 (to drop t_0 or t_1 from the interval would be to render the correspondence discontinuous). Or we write $-\infty < t < \infty$ and identify ∞ and $-\infty$, and thus introduce the singularity ∞. This is no singularity of the curve. It is merely an unhappy lack of correspondence between arithmetic and geometric types of order. H. B. Phillips and C. L. E. Moore in their "Algebra of plane projective geometry," *Proceedings of the American Academy*, vol. 47, pp. 737–790, probably would have been glad to avoid infinity; but with a non-homogeneous algebra that was out of the question.

‡ One of these being euclidean in the cases Professor Bôcher discusses.

If we start with non-homogeneous (rectangular cartesian) coordinates and write down the equations of a general linear transformation, we get introduced to infinity through the vanishing of the denominator which is common to the two fractions. However, this means merely that from the point of view of projective geometry and the projective plane, we have made a very injudicious start—howsoever judicious the start may have been for a modulation of a pedagogical nature from metric to projective geometry for the student first approaching the latter. Had we selected a triangle with a unit point, introduced trilinear coördinates based on a projective number system, and used the homogeneous form of the transformation, such as is almost invariably used in invariant theory, we should never have met any infinity.

We believe in the pedagogic modulation, but we believe also that particular stress should be laid on the fact that the ideal elements which are introduced are ideal elements of the extended euclidean plane and of extended euclidean geometry rather than of projective geometry, and that they are introduced in or, better, adjoined to the euclidean plane for the purpose of bringing about a correspondence (one out of infinitely many) between the euclidean and projective planes, not only for pedagogic purposes, but rather especially for the sake of carrying across theorems from either geometry into the other. The process of throwing some line, a perfectly normal line of the projective plane (we cannot say a finite line because there is no distinction of finite and infinite), into the ideal region of the euclidean plane is of great use in saving a new demonstration of certain theorems.

When we turn to circle geometry a similar state of affairs is found. We are able to set up a correspondence between the euclidean and circular planes which is one-to-one, points corresponding to points and circles to circles (with proper qualifications), except that there is an extra point in the circular plane for real geometry and two extra imaginary lines intersecting in a real point in the case of the complex circular geometry. We promptly adjoin these as ideal elements in the euclidean plane for the sake of perfecting the correspondence—the reasons being as before partly pedagogic, partly lexicographic.*

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* It would indeed be an interesting study in euclidean geometry to take the general solution of the Apollonian problem as developed in inversive

It is somewhat doubtful whether we should regard the ideal regions thus adjoined to the euclidean plane as lying in that plane; it may be better to regard them as lying nowhere at all. Two reasons for this occur to anybody at once. First, the postulates upon which euclidean geometry has been built up are in many cases no longer true for the extended euclidean plane. The introduction of the ideal elements into the euclidean plane has simplified certain statements, namely, such as are essentially projective or inversive (as the case may be), but it has greatly complicated others, which are essentially euclidean. Second, the two cases we have considered show that different ideal regions have to be adjoined in different cases, and that these different ideal regions are mutually incompatible so that they cannot coexist.

The problem of correspondence between the projective and the euclidean planes or between the inversive and euclidean planes is simple by virtue of the fact that the planes do not really clash, in each case we have merely to remedy a defect in the euclidean plane. A vital reason, too, for the naturalness of the correspondence lies in the fact that the euclidean group

$$p, \quad q, \quad yp - xq$$

is a subgroup of the projective group

$$p, \quad q, \quad yp - xq, \quad yp + xq, \quad xp + yq, \quad xp - yq,$$
$$x^2p + xyq, \quad xyp + y^2q,$$

and of the inversive group

$$p, \quad q, \quad yp - xq, \quad xp + yq,$$
$$2xyp + (y^2 - x^2)q, \quad (x^2 - y^2)p + 2xyq.$$

When we try to establish a correspondence between two geometries which are not related in such a manner that one is a subgroup of the other, the matter is not so simple. How could we set about mapping the projective plane on the inversive plane or vice versa? It is futile to content ourselves with trying to map the points of one upon the points of the other; we must be able to carry over certain configurations or

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geometry and see how many special cases we might get by selecting different points of the figure for relegation to infinity, and the Apollonian problem for euclidean geometry is not solved until all possibilities have been enumerated.

relations. But projective geometry deals with lines, conics, and so on, whereas inversive geometry deals with circles, cyclides, and so forth. As a problem in abstract geometry, in logic, it appears somewhat difficult to set up a satisfactory and useful relationship between the planes.*

We may overcome the difficulty very readily by coming down to the common subgroup, to euclidean geometry,† and make the transfer between projective and inversive geometry by the intermediary of euclidean. We notice now, however, that the projective and inversive planes clash in their requirements for extensions of the euclidean plane and we shall be on more comfortable ground if we keep the intermediary instead of attempting to cast it aside and obtain a direct correspondence. Projective geometry can be forced into the inversive mould, or inversive geometry into the projective mould only by cracking the mould or the geometry; but if we insist on making the correspondence direct, it would be difficult to say why we should locate the crack at infinity in either geometry.

The definitions of geometry and of infinite region which we have attributed to Professor Bôcher (Definitions 1 and 2) suggest at once the methods of Sophus Lie, and it is an interesting question to ask whether in plane geometry we are bound to introduce for different groups other regions at infinity than those which arise in the projective and inversive groups. Now Lie has tabulated the varieties of groups which occur in plane geometry in twenty-six entries.‡ We may integrate and determine the finite equations of the group. I have done this in a great many cases and have not found any other types of region.§ In space‖ we have a considerable variety—a plane, region at all, and to modify the postulates so as to introduce any points at infinity would be an unpleasant complication resulting in no gain to the geometry. When, however, we desire to map the projective (or inversive) and Lobachevskian planes one upon the other, we find that the Lobachevskian plane lies entirely within a conic of the projective plane (and entirely upon one side of a circle in the inversive plane—there is here no distinction between inside and outside). To perfect the correspondence we adjoin to the Lobachevskian plane the conic (or circle) as an infinite region and the region outside the conic (or upon the other side of the circle) as an ultra-infinite region.

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* Abstractly a similar difficulty exists in the case of the projective (or inversive) plane and the euclidean plane; for the projective and euclidean lines are different, the inversive and euclidean circles are not the same—but the differences are not so serious, there are marked similarities as well as differences.

† Indeed to what Klein calls the Hauptgruppe, p, q, $yp - xq$, $xp + yq$, which contains similitude transformations; but euclidean geometry is more familiar.

‡ Continuierliche Gruppen, p. 360.

§ This does not mean that in all similar groups, that is, in all reducible to a common type by a proper choice of variables, the infinite region would be the same; it merely means that a choice of variables exists for which the infinite region becomes as indicated.

‖ Transformationsgruppen, vol. 3, pp. 122–178.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, MASS.,
February, 1914.

FAMOUS PROBLEMS OF GEOMETRY.

"*Squaring the Circle,*" *A History of the Problem.* By E. W. HOBSON. Cambridge, at the University Press, 1913. iii+57 pp. Price 3s.

A FASCINATING and voluminous volume could be written on ancient problems of geometry, their influence on the progress of mathematics and the various developments in mathematics which contributed to their generalization or final settlement.

There is the familiar problem,* to draw from a given point P a line such that the line segment cut off by two intersecting lines l_1, l_2 shall be of given length. This problem is capable of solution with ruler and compasses in but one case, namely when P is on a bisector of an angle between l_1 and l_2. Suppose this condition to obtain. The problem is not an easy one, in general, but Apollonius (about 225 B.C.), known to his contemporaries as the "great geometer," found an elegant solution.† The complete discussion for the case of l_1 and l_2 at right two planes intersecting in a line, three planes with a point in common, a quadric cone,* and perhaps other cases.

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* Cf. my paper "Discussion and history of certain geometrical problems of Heraclitus and Apollonius," *Proc. Edinb. Math. Soc.*, vol. 28 (1909–1910), pp. 152–178.

† Although twice proposed in the *American Math. Monthly* (Feb., 1910, vol. 17, p. 48, cf. pp. 140–141; Feb., 1911, vol. 18, p. 44, cf. pp. 114–115) no solution has been forthcoming.

We have only one or two more illustrations to give toward substantiating our point of view that the introduction of infinity as Professor Bôcher does is not entirely satisfactory from the geometric point of view, and indeed violates established nomenclature.

Suppose that we consider the group

$$yp - xq, \quad x^2p + xyq - p, \quad xyp + y^2q - q.$$

This is a projective subgroup which leaves the circle $x^2 + y^2 - 1 = 0$ invariant, and is closely associated with Lobachevskian geometry.† According to ordinary nomenclature the fixed circle is the absolute and the absolute is at infinity. Not so, however, if we must determine the finite equations of the group and see what nature of locus is relegated to the bourne beyond the euclidean plane.‡

Suppose that we consider the group

$$yp - xq, \quad x^2p + xyq + p, \quad xyp + y^2q + q.$$

This is a projective subgroup§ which leaves $x^2 + y^2 + 1 = 0$ invariant, and is closely associated with Riemannian geometry. According to ordinary nomenclature lines in this geometry are closed and have a finite length (at least in the real plane) and there does not arise the question of infinity. Not so, however, if again we must determine the finite equations of the group and observe what manner of locus it is that yields infinite values for the transformed coördinates.

The case of real Lobachevskian geometry is illuminating. When that geometry is defined by a set of postulates (see Coolidge, Non-Euclidean Geometry) there is no infinite

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[*] If we may call the minimum cone in inversive space geometry a quadric cone. Can we appropriately, except in euclidean geometry, call it a minimum cone?

[†] See Klein-Fricke, Automorphe Funktionen. We can also, as Klein has pointed out, use a subgroup of inversive geometry; this is conformal but lines become circles orthogonal to a fundamental circle.

[‡] This line lies in what is called the transfinite or ultra-infinite (as contrasted with finite or infinite) region. See Coolidge, Non-Euclidean Geometry, p. 85.

[§] With an appropriate change of variable, the group may be made a subgroup of the inversive group, as suggested in the second footnote above. Indeed the groups which leave $x^2 + y^2 \pm 1 = 0$ invariant are

$$yp - xq, \quad (x^2 - y^2)p + 2xyq + p, \quad 2xyp + (y^2 - x^2)q \pm q.$$

angles, was known to Heraclitus. When P was not on a bisector, the Greeks solved the problem by means of: (*a*) the conchoid of Nicomedes, and (*b*) an hyperbola. More than a thousand years elapsed; then we find Girard (in 1629) dealing with the Heraclitus case and led to an equation of the fourth degree. He gives geometrical interpretation of negative roots, a notable achievement in his time. A few years later (1637) Descartes considered the same problem more generally. Huygens found many solutions, those which are geometrical of the Apollonian case being of remarkable elegance. Newton with keen penetration showed how, with appropriate choice of unknown, the problem leads to the solution of a quadratic equation, whereas, without suitable choice of unknown, we get a reducible biquadratic. Any complete discussion of the problem must also refer to L'Hospital, Ghetaldi, Gergonne, d'Omerique, Steiner, and to the solution of the general problem through methods of projective geometry by G. Russo.

Then recall the "problem of Apollonius," to describe a circle tangent to three coplanar circles, with its many special cases, how it led to developments in geometry.* In 1600 Viète treated the problem by synthetic methods; analytical solutions were found by Newton, Euler and N. Fuss. Fermat discussed, synthetically, the analogous problems for spheres. In later day a solution of this problem was found by pupils of Monge, who discussed the motion of a variable sphere tangent to three given spheres. Dupuis discovered and Hachette proved (1804) that the middle point of the sphere moves on a conic, and that its points of contact describe circles. Soon after (1813) Dupin published his researches on the remarkable surfaces which envelope such a variable sphere, surfaces which he afterward named cyclides. Then followed the ideas of radical axes, power of circles and spheres, similitude, and inversion, connected with developments of Gaultier, Poncelet, Quetelet, Steiner, Plücker, Magnus, Ingram, and Stubbs.

Again, there is the less known problem of Pappus, to inscribe in a circle a triangle whose sides (or sides produced) shall pass through three given points on a line. This is solved by Pappus as a lemma useful for attacking the problem of Apollonius (in just what way is an interesting speculation).†

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* Cf. Reye, Synthetische Geometrie der Kugel und linearer Kugelsysteme. Leipzig, 1879.

† Cf. my paper entitled "Centers of similitude of circles and certain theorems attributed to Monge. Were they known to the Greeks?", which is soon to appear in the *Amer. Math. Monthly.*

The generalization of the problem to the case where the given points were any points was found by Simson (1731). After thirty years of attention Castillon gave a solution to the Berlin Academy in 1776. Lagrange, who was present at the meeting, handed a trigonometric solution to Castillon on the next day "apparently for the purpose of showing that an analyst could solve in a single evening a problem that had confessedly required from a geometer of no mean powers many years of study." Then Lexell, Euler, and Fuss discussed the problem but no method was found which could be generalized so that a polygon could be substituted for a triangle. This method was discovered by Giordano di Ottajano, a student at Naples in 1784, when only 16 years of age. Malfatti, Romanus, Lhulier, Carnot, and Meyer Hirsch made later contributions to the discussion. William Wallace, famous for his discovery of Wallace's line (incorrectly called Simson's line), first pointed out (in 1798) indeterminate cases in the triangle problem. To follow up developments in the English school: Lowry made obvious generalization of the problem (in 1806) to a polygon inscribed in an ellipse; Hearn considered the general dual problem of circumscription (instead of inscription) due to Gergonne; and then there are researches of Townsend, Potts, and Renshaw leading up to results of Sir William Rowan Hamilton regarding polygons inscribed in a sphere, an ellipsoid and an hyperboloid, and with sides passing through given points.* But in the early part of the nineteenth century in the French school we find that Gergonne, Servois, Poncelet, Brianchon, and others discussed various phases of the problem of polygons in conics, some of them involving the newly discovered principle of duality.

Similar summary sketches of many other notable problems, with like origin, might be given; among these the most famous are the three problems of the ancients, known under the titles: the duplication of the cube, the trisection of an angle,† and the squaring of the circle. Although all of these are somewhat special in character, they have, as Hobson remarks, "one great advantage for the purposes of historical study, viz. that their complete history as scientific problems lies, in completed form, before us." Taking the last of these problems, Hobson continues, "we possess indications of its origin in remote antiquity, we are able to follow the lines on which the treatment of the problem proceeded and changed from age to age in accordance with the progressive development of general mathematical science, on which it exercised a noticeable reaction. We are able to see how the progress of endeavors towards a solution was affected by the intervention of some of the greatest mathematical thinkers that the world has seen, such as Archimedes, Huygens, Euler, and Hermite. Lastly, we know when and how the resources of modern mathematical science became sufficiently powerful to make possible that resolution of the problem, which, although negative, in that the impossibility of the problem subject to the implied restriction was proved, is far from being a mere negation, in that the true grounds of the impossibility have been set forth with a finality and completeness which is somewhat rare in the history of science."

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* Hamilton considered the solution of these problems which depended on a linear equation in finite differences as an especially tough piece of work. Cf. Life of Sir Wm. R. Hamilton, by R. P. Graves, vol. 3 (1889), pp. 88, 426.

† Who first proved the impossibility of the classic problem of trisection of an angle? I have not met with a statement of this fact in any of the mathematical histories, but surely it was before 1852, when Sir William Rowan Hamilton wrote to De Morgan: "Are you *sure* that it is impossible to trisect the angle by Euclid? I have not to lament a single hour thrown away on the attempt, but fancy that it is rather a tact, a feeling, than a

"If the question be raised, why such an apparently special problem, as that of the quadrature of the circle, is deserving of the sustained interest which has attached to it, and which it still possesses, the answer is only to be found in a scrutiny of the history of the problem, and especially in the closeness of the connection of that history with the general history of mathematical science. It would be difficult to select another special problem, an account of the history of which would afford so good an opportunity of obtaining a glimpse of so many of the main phases of the development of general mathematics; and it is for this reason, even more than on account of the intrinsic interest of the problem, that I have

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proof, which makes us think that the thing cannot be done. No doubt we are influenced by the cubic form of the algebraic equation. But would Gauss's inscription of the regular polygon of seventeen sides have seemed, a century ago, much less an impossible thing, by line and circle?"

De Morgan replied: "As to the trisection of the angle, Gauss's discovery increases my disbelief in its possibility. When $x^{17} - 1$ is separated into quadratic factors, we see how a construction by circles may tell. But, it being granted $ax^3 + bx^2 + cx + d$ is *not* separable into a real quadratic and a linear factor, I cannot imagine how a set of intersections of circles can possibly give no more or less than three distinct points." Graves' Life of Sir Wm. R. Hamilton, vol. 3 (1889), pp. 433–435.

selected it as appropriate for treatment in a short course of lectures."

Thus Professor Hobson gave six lectures at Cambridge University during the Easter term of last year, on the history of the problem of the quadrature of the circle. These lectures now appear in print in the volume before us.

It is 160 years since Montucla published anonymously his little book[*] on the history of this problem. In 1831, again in anonymous fashion, it was revised and corrected by Lacroix. Some sixty years later came Rudio's elaborate and excellent work.[†] Then there have been the Bändchen of Schubert[‡] and of Beutel.[§] Other historical surveys, not published in separate volumes, were written by T. Muir,[||] B. Calò,[¶] Cantor,[**] T. Vahlen[††] and D. E. Smith.[‡‡]

Professor Hobson's sketch is divided, as Rudio's, into four chapters: one a general account of the problem, and one for each of three historical periods "marked out by fundamentally distinct differences in respect of method, of immediate aims, and of equipment in the possession of intellectual tools."

Introductory to the general survey we are reminded that from the time of the Greeks down to our own day the problem has been very widely known outside of mathematical circles and many such people have occupied themselves in seeking a solution. In fact, as Schubert remarks, the Greeks had a special word to designate this species of activity, namely *τετραγωνιζειν*, which means to busy one's self with the quadrature. In later day, among the scientific cranks with whom almost every scientific society has had to deal, is the insistent circle squarer amusingly dissected by De Morgan in his Budget of Paradoxes. So pestered were the members of the French Academy that as far back as 1775 they passed a resolution not to examine, from that time on, any so-called solutions of the quadrature of the circle that might be handed in.

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[*] Histoire des recherches sur la Quadrature du cercle . . . avec une addition concernant les problemes de la duplication du cube et de la trisection de l'angle, Paris, 1754, pp. xliii+304+8 pls.

[†] Archimedes, Huygens, Lambert, Legendre. Vier Abhandlungen über Kreismessung . . . mit einer Übersicht über die Geschichte des Problemes von der Quadratur des Zirkels, Leipzig, 1892, pp. viii+166.

[‡] H Schubert, Die Quadratur des Kreises in berufenen und unberufenen Köpfen. In Virchow-Holtzendorffs Sammlung, Heft 67, Hamburg, 1889. Translated in *Monist*, Jan., 1891, vol. 1, pp. 197–228; reprinted in *Smithsonian Report* for 1890, pp. 97–120.

[§] E. Beutel, Die Quadratur des Kreises (Mathematische Bibliothek, Nr. 12), Leipzig und Berlin, 1913, pp. 75.

[||] Encyclopaedia Britannica, ninth edition, article "Squaring of the circle," 1887; eleventh edition, article "Circle," 1910.

[¶] In Enriques's Fragen derElementargeometrie, 2. Teil, Leipzig u. Berlin, 1911.

[**] Vorlesungen uber Geschichte der Mathematik, vol. 1, 3. Aufl., 1907; vol. 2, 2. Aufl., 1900, reprinted 1913; vol. 3, 2. Aufl., 1901; vol. 4, 1908.

[††] Konstruktionen und Approximationen, Leipzig u. Berlin, 1911.

[‡‡] In Monographs on Topics of Modern Mathematics, ed. by J. W. A. Young, New York, 1911.

On pages 4–10 Hobson sets forth the implications of the problem and notes that the fact was well known to the Greeks that the problems of the quadrature and rectification of the circle were really equivalent. He emphasizes assumptions and limitations of euclidean constructions and remarks, without direct reference to Mascheroni, that all such are possible with compasses alone. His discussion is then summed up: As an ideal problem the quadrature of the circle "is capable of accuracy bounded only by the instruments employed. Ideally we can actually determine, by euclidean methods, a square of which the area differs from that of a given circle by less than an arbitrarily prescribed magnitude, although we cannot pass to the limit."

The first historical period commences with the early empirical determinations of the ratio of the circumference of a circle to its diameter, and ends with the invention of the calculus in the latter part of the seventeenth century. This period was characterized by geometric discussion, for example, the method of exhaustions coupled with polygons circumscribed and inscribed. The first determination of the value of π is to be found in the Papyrus Rhind preserved in fragmentary condition in the British Museum and now being translated into English by Chancellor and Mrs. Chace of Brown University. This papyrus, written about 1700 B.C., is based upon a much earlier work. We here find $\pi = 256/81 = 3.1604 \cdots$. From the time of Chou-Kong, who lived in the twelfth century B.C., Chinese mathematicians employed the approximation $\pi = 3$. This value was also known to the Babylonians and is implied in verses of the Bible in Kings and Chronicles which are believed to have been written in the sixth and fifth centuries B.C. About 420 B.C. Hippias of Elis invented, for the solution of the problem, a curve known as the *τετραγωνιζουσα* or quadratrix, and about the same time Hippocrates of Chios developed his theory of menisci or lunulæ. But the first really scientific treatment was given by Archimedes

(287–212 B.C.). Considering circumscribed and inscribed polygons of 96 sides, he was led to the result $3\frac{1}{7} < \pi < 3\frac{10}{71}$ that is $3.14285 \cdots < \pi < 3.14084 \cdots$. The next more exact determination was found by Ptolemy (87–165 A.D.) to be $3\frac{17}{120} = 3.14166 \cdots$. Four centuries later the great Chinese astronomer Tsu Ch'ung-chih proved that 10π lies between 31.415927 and 31.415926 and thence deduced the value $\frac{355}{113}$,* which is correct to six places of decimals. François Viète (1540–1603) obtained the value of π correct to 9 places, Adrianus Romanus to 15, Ludolph van Ceulen to 35. Huygens, Descartes, and Gregory introduced new methods of discussing the problem. Of approximate euclidean solutions may be mentioned that of J. de Gellert (*Grunert's Archiv*, 1849) which is readily deduced from the fact that $\frac{355}{113} = 3 + \frac{4^2}{7^2+8^2}$. The integraph, invented and described by the Russian engineer, Abdank-Abakanowicz, and its application to the construction of π, are nowhere mentioned by Hobson.

The second period, from the second half of the seventeenth to the latter part of the eighteenth century, is characterized by analytic discussion of the problem. J. Wallis, Lord Brouncker, Gregory, Machin, Leibnitz, Euler, and others obtained various expressions for π involving inverse tangents of numerical quantities, definite integrals, infinite products and series. Machin's relation

$$\frac{\pi}{4} = 4 \tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

was used by Shanks who gave, in 1873, the value of π correct to 707 places of decimals. Professor Hobson dwells on Euler's work, for it is recalled that the final settlement of the problem was based upon the remarkable relation which he discovered,† namely

$$e^{i\pi} + 1 = 0.$$

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* This is very easily remembered by writing down the first three odd numbers, each repeated as 113355, and recalling that the first three digits are to be placed in the denominator, the last three in the numerator.

† The form $e^{ix} = \cos x + i \sin x$ was first given by Euler in *Miscellanea Berolinensia*, vol. 7 (1743), p. 179; paper read Sept. 6, 1742. He used the symbol e in a letter dated November 25, 1731 (Corresp. Mat. et Phys., par Fuss, vol. 1 (1843), p. 58. He employed i for $\sqrt{-1}$ in 1777 (Institutionum calculi integralis, vol. 4 (1794), p. 184: "De formulis differentialibus . . .," presented to the Akademie, May 5, 1777). It should be noted

Although in this period great progress was made toward the solution of the problem, the true nature of the number π had not yet been discussed; but from the end of the seventeenth century none of the prominent mathematicians doubted that π was irrational.*

The third period commences with the proof of this fact by J. H. Lambert (1728–1777). Rudio's statement that this proof "fehlte zur völligen Strenge" has been copied without checking by Klein, Smith, and others, but Pringsheim showed that Rudio was in error. Following him, Vahlen and Hobson state the matter correctly. Nearly 80 years were to elapse before Liouville proved the existence of transcendental numbers. Hobson reproduces the 1851 presentation, in the main. Not till 1873 did Hermite prove the transcendence of e. Finally in 1882 Lindemann succeeded in establishing the transcendence of π and the problem whose history covers 4000 years was finally solved. Simplifications of Lindemann's proof soon followed, and one of these by Gordan is the basis of the condensed four-page Hobson proof, which requires knowledge of the elements of the differential calculus. It is assumed on page 55 that there are an infinite number of prime numbers. This was known to Euclid (Elements XI, 20.)

On page 50 the author establishes the theorem: "In order that a point P may be determinable by euclidean procedure it is necessary and sufficient that each of its coordinates be a root of an equation of some degree, a power of 2, of which the coefficients are rational functions of $(a_1, a_2, \cdots, a_{2r})$, the coordinates of the points given in the data of the problem." Hobson starts with

$$\begin{aligned} (1)\qquad x = a + b\sqrt{c_1 \pm \sqrt{c_2 \pm \sqrt{c_3 + \cdots}}} \\ + b'\sqrt{c_1' \pm \sqrt{c_2' \pm \sqrt{c_3' + \cdots}}} + \cdots . \end{aligned}$$

But it is not shown that the equation referred to in the theorem may not have other roots than those given by (1). Multiple

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that Roger Cotes gave the formula $ix = \log(\cos x + i \sin x)$ (*Phil. Trans.*, 1714), which is practically the same as the above as we look at it now, long before Euler.

* Euler, Introductio in Analysin Infinitorum, vol. 1 (1748), p. 93. It would be interesting to have Professor Hobson's authority for the statement that Euler gave expression to the conviction "that e and π are not roots of algebraic equations."

roots derived from (1) are not considered either, and the term "irreducible equation" is not mentioned. For this discussion as well as for the proof of the transcendence of π, the student will probably find more satisfactory and more elementary treatment in Klein's remarkable little book, Famous Problems in Elementary Geometry,* to which our author nowhere refers.

Professor Hobson's most interesting and attractively written book is such as to fire the imagination of a young student of mathematics and as to impel him to more persistent and enthusiastic effort. It is sure to have a wide circulation. So much the more then is it to be regretted that lamentable lack of care is displayed in the proofreading and final checking of details, by the author. His authorities, we are informed, included Heath,† Rudio, Cantor, Mikami,‡ and Vahlen, who can rarely, if ever, be blamed for the errors committed. Against an historical work coming from such a source, these are serious charges. It may therefore be well to support them at some length.

Obvious slips in typography occur on page 46, lines 3 and 10, page 55, line 18, in the exponent of C. On page 55, lines 25, 27, 30, with usual notation for differentials, brackets should surround the exponents of ϕ. In mathematical writing, uniformity seems highly desirable, at least within the limits of a 60 page book: why then the proper names in italics on pages 23, 25, 27, and why the three forms Leonardo Pisano, Nicholas of Cusa, Leonardo da Vinci? Three different methods are used for referring to the volume of a periodical work (for example, page 31, and page 44, lines 6 and 9) while one of these methods is elsewhere used for a book in a volume (page 31, line 29).

Before the abbreviation "Kl." on page 44 and again in a footnote on page 53, "Math.-nat." should be placed. Many names are spelled incorrectly: In seven places an extra *h* is put in Huygens, Raaus should be Ra-ā-us', for Áryabhatta read Âryabhaṭṭa; for Chang Hîng, Chang Hêng; for Wang Fau, Wang Fan; for Gancea, Ganeça; for Alchwarizmi, Alchwarizmî; for George Purbach, Georg von Peurbach

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[*] Boston, 1897; German ed., Leipzig, 1895; French ed., Paris, 1896; Italian ed., Torino, 1896; Russian ed., Kasan, 1898.

[†] Works of Archimedes, Cambridge, 1897.

[‡] The Development of Mathematics in China and Japan, Leipzig, 1912.

or Georg of Peurbach; for Rheticus, Rhäticus; for Ludolf van Ceulen, Ludolph van Ceulen; for Willebrod Snellius, Willebrord Snellius; for Grünert, Grunert; for Tanyem Shŏkei, Tanzan Shōkei.

We are told that Professor Landau discussed lunulæ in 1890, that is when he was 13 years of age. This date should be 1902, and the reference to "Ueber quadrierbare Kreisbogenzweiecke," *Sitzungsberichte der Berliner Mathematischen Gesellschaft*, volume 2 (1902), pages 1–6. Three dates of birth are given incorrectly: Ludolph van Ceulen, 1539 instead of 1540 (N.S.); Newton, 1642 instead of 1643 (N.S.); Snellius, 1580 instead of 1581. The phrase "is said to have," page 27, line 4, should be deleted and after the next sentence, "According to his wish the value was engraved on his tombstone which has been lost," should come: "but the inscription has been preserved (Bierens de Haan, *Messenger of Mathematics*, volume 3 (1874), page 25)."

Vega calculated π correctly to 136 not 140 places.[*] Two slips occur in connection with W. Shanks' name (page 39): (1) he published his value of π to 530 places in *Proceedings of the Royal Society*, 1853–54, but his value to 607 places in Contributions to Mathematics, London, 1853, pages 86–87; (2) he gave 707 places in 1873, not "1873–74." On page 40 for "1766" read "1769," the year when the result was printed; and on page 39 for "1755" read "1779 (though not published till 1798);"[†] for "The same series was also discovered independently by Ch. Hutton" read "The same series was first discovered by Ch. Hutton."

The statement on page 41: "He [Euler] introduced the practice of denoting each of the sides and angles of a triangle by a single letter," is inaccurate, as Caswell[‡] (and Oughtred) had already named each of the sides by a letter. Hobson might have correctly written: "Euler was one of the first if not the first to introduce the notation *a*, *b*, *c*, for the sides opposite the angles A, B, C, of a triangle, etc."[*] And again, did Euler introduce "the short designation of the trigonometrical ratios by $\sin \alpha$, $\cos \alpha$, $\tan \alpha$, etc."? The originals of the remarkable memoirs of 1753 and 1781[†] are not available to me, but in E. Hammer's translations[‡] "tan" never occurs; it is always "tg"; so also "ctg" not "cot"; furthermore the abbreviation "cosec" or "csc" does not come in at all. In 1777 J. T. Mayer used the following abbreviations:[§] sin, cos, tang, cot, sec, cosec.

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[*] G. Vega, Thesaurus Logarithmorum, Leipzig, 1794, p. 633, and W. Shanks, Contributions to Mathematics, London, 1853, p. 86.

[†] "Conventuri exhibita die 7 Junii, 1779" *Nova Acta Acad. sc. Petrop.*, vol. 11 (1793), 1798, p. 133. Cf. G. Eneström, Verzeichnis der Schriften Leonhard Eulers, Leipzig, 1910, p. 181.

[‡] "Suppose m, n the legs of the angle required; B its base . . . $\zeta = \frac{1}{2}$ sum m, n, B," page 6 of A Brief (but full) Account of the Doctrine of Trigonometry Both Plane and Spherical, by John Caswell, London, 1685; bound in with Wallis's Treatise of Algebra, London, 1685.

Professor Hobson recalls that "the notation π appears as early as 1706, when it was used by William Jones in Synopsis palmariorum matheseos." It would have been of interest to have further remarked that the notation was probably suggested to Jones by Oughtred[||], whose name stands out much more prominently in English mathematics. Liouville's papers on transcendental numbers are dated 1844[¶] and 1851,[**] not 1840 (page 44). To the sentence: "The simpler of Liouville's methods of proving the existence of such numbers will be given here" should be added, "with some alterations." And finally Professor Hobson seems to have forgotten (page 53) that it was the third, not the second, edition of his Treatise on Trigonometry which was published in 1911.

In broad outline the work is accurate; but when the mathematician writes now-a-days, he is expected to have a much higher ideal. As clearly stated in the preface, no original contribution to the history of the subject is attempted; for this we must await the forthcoming edition of Pappus by

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[*] This was done in the case of *spherical* triangles in 1753: "Principes de la trigonométrie sphérique tirés de la méthode des plus grands et plus petits," *Mém. de l'Acad. d. Sc. de Berlin*, vol. 9 (1753), 1755, pp. 223–257.

[†] *Acta Acad. Sc. Petrop.*, 1779; I, printed 1782, pp. 72–86. "Nach den Akten am 12 März, 1781 der Petersburg Akademie vorgelegt." (G. Eneström, Verzeichnis der Schriften L. Eulers, Leipzig, 1910, p. 149.)

[‡] Ostwald's Klassiker der exakten Wissenschaften, No. 73, Leipzig, 1896.

[§] Gründlicher und ausführlicher Unterricht zur praktischen Geometrie, Erster Theil, Göttingen, 1777, p. 5 ff.

[||] William Oughtred (1574–1660) in his Clavis Mathematica of 1631, etc., and in his Theorematum in libris Archimedes de Sphaera et Cylindro Declaratio, Oxford, 1652, frequently employs the symbol $\delta:\pi$ or $\pi:\delta$ (in modern rotation) for the ratio of the semi-diameter to the semi-periphery or of semi-periphery to semi-diameter. It is noticeable that these letters are *never* used separately, that is, π is *not* used for "Semiperipheria," as Tropfke suggests (Geschichte der Elementar-Mathematik, vol. 2 (1903), p. 135). Oughtred states specifically in his "Theorematum": "$\pi R/\delta$, est semiperipheria circuli cujus Radius est R."

[¶] *Comptes Rendus*, vol. 18, pp. 883, 910.

[**] *Liouville's Journal*, vol. 16, pp. 134, 139.

Professor Hobson's fellow countryman, Sir Thomas L. Heath. It is certainly not well to overload a work of this kind and suggestions for filling up the many lacunae are hardly in order. It appears to the reviewer, however, that the value and usefulness of the work would be considerably enhanced if many more exact references were given to original sources as authorities for the statements that are made, and if an index of names were appended.

In conclusion, and in illustration of the statement made above, that others beside mathematicians were cognizant of the problem of squaring the circle, it may be of interest to put on record two references to literary classics. In the Birds (produced in 414 B.C.) of Aristophanes there is a long dialogue between Peisthetaeros and Meton, the great astronomer, introduced as a solemn quack talking unintelligible nonsense for the most part. In lines 1004–5, however, Meton is made to say:*

"With the straight rod I measure out, that so
The circle may be squared."

The second reference takes us to the early fourteenth century. It is to the last canto of the Paradiso where Dante compares his inability to penetrate by his own unaided power the mystery of the Incarnation, to that of one

"Who versed in geometric lore, would fain
Measure the circle;† and though pondering long
And deeply, that beginning, which he needs,
Finds not; . . . ."‡

Are there references to other passages of this kind before 1500? One may turn to the great Oxford Dictionary for those of later date.

R. C. ARCHIBALD.

BROWN UNIVERSITY,
PROVIDENCE, R. I.,
January 30, 1914.

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* Translation of B. B. Rogers, London, 1906.

† "Misurar lo cerchio;" Longfellow translates this "To square the circle."

‡ Cary's translation, Canto 33, lines 133–135.

SHORTER NOTICES.

New Analytic Geometry. By PERCY F. SMITH and ARTHUR SULLIVAN GALE. Boston, Ginn and Company, 1912. x+342 pp.

AT the present time when there are so many text-books which treat of analytic geometry, a new book in order to justify its publication should be original both in its methods and in the material which it contains. If this is taken as our standard, Smith and Gale's New Analytic Geometry ought certainly to find a place among the modern college text-books of elementary mathematics. The authors have succeeded in getting away from the beaten and orthodox path followed by a majority of similar text-books and have given to the subject a treatment which is in harmony with progressive ideals in college mathematics.

In the selection of material the book is unique, if we except with some reservation a few other books such as Lambert's Analytic Geometry or Smith and Gale's Introductory Course in Analytic Geometry. In the table of contents are to be seen such chapter headings as Transcendental curves and equations, Functions and graphs, Parametric equations and loci, Empirical equations, etc. A hasty glance is sufficient to give the impression that the Analytic Geometry is not a treatise on conic sections, the discussion of that important class of curves being limited to one short chapter of thirty-five pages. The authors have emphasized the fact that they are presenting a general treatment of loci problems, and that corresponding equations can be determined and interpreted regardless of their forms or of the kinds of coordinates which are used. The aim seems to have been not to study intensively a limited class of curves and equations but rather to develop general laws which can be applied equally well to all parts of the field.

The teacher of calculus and of more advanced mathematics is naturally pleased to see the new chapters on transcendental curves and parametric equations as well as an excellent chapter on polar coordinates. In these parts of the book are found a great many of the typical equations and curves which are used in illustrations in calculus and which appear very frequently in technical work. More space has been given to transcendental and parametric equations than to the equations of the conic sections. It must, however, be admitted that, from the standpoint of the student of technology, this division of space can be justified.

A chapter on Empirical equations is, for a text-book of analytical geometry, decidedly an innovation. However, the treatment is far from adequate and lacks almost completely any evidence of mathematical rigor. Such a chapter, however, may be the means of stimulating the reader to investigate more carefully this important field of applied mathematics. It is exceeding doubtful whether any treatment of the subject of curve fitting can be so condensed and simplified as to find a place in a text-book of analytic geometry and at the same time be adequate from a strictly mathematical point of view.

Ninety-seven pages of the text are given over to the analytical geometry of space. This part of the book, however, seems to lack much of the originality and freshness which characterizes the rest of the book. If the treatment is superior to the same parts of other text-books it is due to the most excellent list of problems which are included.

In their presentation of the subject the authors have used a modified form of the euclidean method. The important theorems are formally stated and the formal proof ending with the usual Q. E. D. follows. Among these important theorems are found those for the distance between two fixed points, the slope-intercept equation of the straight line, the angle between two lines, the equation of a circle, etc. Other theorems are presented as problems. The problem is stated and the solution given, then the result of the solution is summarized in a theorem or in a set of general rules to be used in solving specific problems of the same general character. A third method of presentation is also used which makes use of an illustration. For example, the method of deriving the equation of the tangent at any point on a curve (see page 190) is developed by means of a solution of the problem for the cubical parabola. While this illustrative method may be criticized because of its lack of rigor and generality, it is certainly quite often much more intelligible to the average student than a long and abstract presentation of the same thing. By using these three ways of presentation the strict euclidean method, which was used in the former treatise by the same authors, has lost much of its harshness.

The mechanical side of the book has been splendidly executed and is almost entirely free from errors. The use of italics and heavy type is not overdone, but is still sufficient to emphasize the really important things of the text. The figures are abundant and well drawn. The problems are numerous and have been selected because they illustrate something, rather than because of their difficulties.

Taken as a whole, the book is thoroughly up to date and well written. Although it is more of a drill book than a treatise, it ought nevertheless to furnish the student with a good foundation for a later course in calculus. It is a book which should be a stimulus to every teacher who is in sympathy with the international movement towards improvement in the curriculum and the methods of instruction in college mathematics.

EDWIN R. SMITH.

Technical Trigonometry. By HORACE WILMER MARSH. New York, John Wiley and Sons, 1914. x+232 pp.

THIS book gives a clear and usable knowledge of the trigonometry underlying the industrial and technical studies. The first chapter is devoted to an explanation of logarithms, while the second and third chapters are given to the solution of right and oblique triangles with their applications. It is in the applications that this book differs from the traditional trigonometries. The exercises are chosen from engineering, physics, manufacturing, etc. A few of the types of problems given are: equilibrium on an inclined plane, bevel gears, spiral gear cutter, dovetail joints, two point ball bearing, nuts, flange angles, tangent galvanometer, track turnout, roof truss, length of belts, concrete stand pipes, sewer construction. Each problem is accompanied by a very well executed drawing. These three chapters cover 186 pages or about 9/11 of the book. Many technical terms are used and explained.

The fourth chapter (13 pages) deals with the functions of the sum and difference of two angles and the functions of multiple angles. The proofs of most of the formulas are left as exercises. No other exercises such as proving identities, solving trigonometric equations, solving practical problems, etc., are given. In the last chapter, namely chapter five, the slide rule is explained. At the end of the book are found tables on length, area, volume, weight, decimal equivalents of parts of an inch, U. S. and metric equivalents and wire gauge sizes.

No tables of logarithms or natural functions are given. Radian measure, identities, trigonometric equations and answers to the exercises have been omitted.

F. M. MORGAN.

College Algebra. By W. B. FITE. Boston, Heath, 1913.

A CHARACTERISTIC tendency in mathematics during the last fifty years or more has been in the direction of a more severe criticism of the proofs of theorems. Hence, Professor Fite is adopting the modern point of view when he aims "to present the elementary principles of algebra in a simple and direct way and to give a rigorous proof of the theorems used." But just at this point one of the real dangers in the teaching of elementary mathematics arises. Those who are interested in emphasizing rigor are likely to be sidetracked by minutiæ to such an extent that the subject as presented by them appears too abstract to the beginner, to whom concreteness in any subject is an absolute necessity. With these facts in mind one might say that the aim of the modern text book in algebra should be to present the subject as concretely as possible and at the same time to develop the critical attitude in the mind of the student. I shall examine Professor Fite's College Algebra chiefly along these lines.

First of all it should be noted that the book is meant to be a *college* algebra. Although all of the subjects treated occur in most of the texts which are used in secondary schools it seems to me that some of the subjects, notably the analytic geometry introduced, belong exclusively to college work. This does not mean that the graph and graphical methods should not be used in secondary work; they should be, by all means. Also, the problems which have been selected are largely new and very attractive, and are of sufficient difficulty to make the average freshman think.

Throughout the book the author has made extensive use of graphical methods and has evidently tried to appeal to intuition and to make the subject tangible by practical illustrations of the principles involved. To this end the problem of finding the equation of a circle through three points is introduced as an application of the solution of simultaneous linear equations in three variables. This is better than to refer to three planes, which is always somewhat unsatisfactory at this period of study. A great many more problems from geometry appear than one usually finds in such texts. The same might be said of physical problems, for which an appendix has been added concisely explaining the fundamental laws of physics. The author does not hesitate to state facts which can be ascertained only by means of higher mathematics to be used as the basis of an algebraic problem; this, to my mind, is an excellent method of encouraging the thoughtful student to pursue the study of mathematics further. To the student who can handle the formal work of the book and understand the problems, algebra should be a tool by means of which certain kinds of information might be obtained.

The importance of rigor is never forgotten. In an early chapter the operations which may be legitimately used in the solution of equations are stated and illustrated. The chapter on mathematical induction—a difficult topic to teach successfully—is admirable in this respect. In the proof of the remainder theorem in §142 that the two sides of the expression are *identically* equal is carefully noted; some texts seem to forget the importance of this step. Another evidence of this spirit of precision appears at the bottom of page 177 and need only be mentioned here. Several proofs are very properly omitted; for instance, the proof that every polynomial in x has at least one root is merely referred to historically; the proof of De Moivre's theorem is left to the student; but all theorems which are proved at all are *rigorously* proved, and wherever it is necessary critical observations are added.

Among other features of the book may be mentioned the chapter on complex numbers. The presentation of this subject is somewhat new and very attractive. The chapter on Inequalities contains much which is useful and which does not appear in any other modern text with which I am familiar. In the proof of theorems by mathematical induction illustrations are given which show convincingly the necessity of both parts of the proof.

I have the following adverse criticism to offer. An exceedingly long and complicated sentence for a text book occurs at the bottom of page 22, the purpose of which could have been accomplished better by several more illustrative examples. The subject of undetermined coefficients could have been briefly explained; the omission of this powerful instrument of algebra to make place for so much geometry is hardly justifiable in a text of this kind. Possibly it would have been of advantage to pay more attention to the theory of the quadratic equation—a subject which so many freshmen have not mastered. On page 155, line 4, the word "inequalities" should be changed to "equalities."

Finally, it is my opinion that the book is distinctly a college text; that the subject has been concretely presented, and that an effort has been made to awaken the spirit of critical revision in the mind of the student.

JOSEPH EUGENE ROWE.

Das Problem der Kreisteilung. Ein Beitrag zur Geschichte seiner Entwicklung. Von DR. ARTHUR MITZSCHERLING. Leipzig, Teubner, 1913. vi + 214 pp. M 7.

THIS book falls naturally into three sections, treating the following topics: (*a*) the division of the circle into equal parts, (*b*) the trisection of angles, (*c*) the polysection of angles. Each section is written from the historical point of view and contains for its general topic an account of the relevant geometric constructions, both exact and approximate, and of instruments by means of which the corresponding practical constructions may readily be made.

The book will be of interest to those who desire an elementary historical account of the topics treated.

R. D. CARMICHAEL.

Allgemeine Theorie der Raumkurven und Flächen. Von V. KOMMERELL und K. KOMMERELL. (Sammlung Schubert XXIX and XLIV.) Zweite Auflage, I. Band, 1909, pp. viii+172; II. Band, 1911, pp. vi+188. G. J. Göschen'sche Verlagshandlung, Leipzig. Price 4.80+5.80 marks.

WITHIN the last two or three years, new works, or old works in new editions, have accumulated rapidly in the field of differential geometry. For do we not have Bell, Darboux, Demartres, Forsyth, Knoblauch, Kommerell, Lilienthal, Salmon, Scheffers, and Smith? There are many American colleges, no doubt, where lecture courses, and texts in a foreign language, are deemed inadvisable. In such places the teacher who offers an introductory course in differential geometry is distinctly at a loss. Bell, Frost, Salmon, and

Smith will not, probably, appeal to any American professor as suitable texts in this connection, whilst the exposition of Eisenhart and of Forsyth is too detailed or is otherwise inadequate. Indeed, only one suitable work is available. That is Goursat-Hedrick's Analysis, volume I.* With portions of this work somewhat amplified, with the English edition of Gauss's famous memoir published by the Princeton University Library, and with selections from Smith, a satisfactory course may be worked up. An appropriate single volume in English is therefore much to be desired. We need such elementary books as Raffy, with its clear-cut, elegant and rigorous demonstrations, or this new edition of Kommerell with theory presented in simple form interspersed with ample problem illustration, while authoritative comment† on the historical development stimulates the interest.

The new edition of Kommerell differs widely from the old, which appeared in 1903. There are corrections, freshly drawn figures, expansions, new paragraphs, rearrangements, new illustrations, new problems, and the elaboration of the second volume of 218 pages into one of 194 and another of 177. The original work of 350 pages has been increased in size by 200 pages, to form a treatise of about double the size of Raffy.

To the first volume only 28 pages were added and, as before, it is devoted to the discussion of space curves, and of surfaces whose equations are in the form $F(x, y, z) = 0$. The second volume originally contained two sections (each of which is now treated in a separate volume): the first (118 pages) on surfaces whose equations are in parametric form, the second (94 pages) on special surfaces (W, minimal, of constant curvature, ruled, triply orthogonal) and congruences.‡

For use in the field indicated, the volumes can be heartily recommended.

R. C. ARCHIBALD.

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* Boston, 1904. A new edition in two volumes may be expected in a year or so.

† Dr. V. Kommerell was the author of "Analytische Geometrie der Ebene und des Raumes" in Cantor's Vorlesungen uber Geschichte der Mathematik, Bd. 4 (1908), pp. 453–576.

‡ The new volume on this subject is entitled: Spezielle Flächen und Theorie der Strahlensysteme, Leipzig, 1911, and was reviewed in this BULLETIN, vol. 19 (1913), pp. 253–4, by Professor Cowley.

Beiträge zu einzelnen Fragen der höheren Potentialtheorie. Von E. R. NEUMANN. (Preisschriften gekrönt und herausgegeben von der Fürstlich Jablonowskischen Gesellschaft zu Leipzig, No. 41.) Leipzig, Teubner, 1912. xxiv+188 pp.

THIS prize memoir is concerned with the two fundamental problems of the potential theory, viz., the determination of a solution V of Laplace's equation satisfying certain continuity conditions either in the domain inside or in that outside the boundary when, for every boundary point s, either the value of V or the derivative $\partial V/\partial n_s$ along the boundary normal n_s is given (first and second boundary problem, respectively).

For the first problem, the method of Green's function gives

$$(1) \qquad V_p = \int f_s \psi_{sp} ds,$$

where f_s is the given boundary value, and ψ_{sp} a function depending on the boundary point s and the variable point p only. The researches of Carl Neumann have suggested the possibility of representing ψ_{sp} as the potential of a mass distribution *on the boundary*

$$(2) \qquad \psi_{sp} = \int \Delta_{s\sigma} T_{\sigma p} d\sigma,$$

where $T_{\sigma p} = 1/r_{\sigma p}$ for the three-dimensional and $= \log r_{\sigma p}$ for the two-dimensional problem, and $\Delta_{s\sigma}$ depends only on the two boundary points s and σ. The investigation of this "fundamental mass distribution" $\Delta_{s\sigma}$ is the main object of the memoir under review.

The method is briefly the following: expressing ψ_{sp} by the infinite series furnished by Carl Neumann's method for solving the first boundary problem, this series is divided into two parts. The first consists of a finite number of terms and contains all the singularities of ψ_{sp}, while the second part, consisting of the remainder of the series, has no singularities. It is found that the representation (2) is possible for the second part, but not for the first, which, however, is of sufficient simplicity in itself.

Making $h = 2$ for the three-dimensional and $h = 1$ for the two-dimensional problem, the following formulas are obtained, according as the variable point p is an interior point i or an exterior point e:

$$(3)\quad \begin{aligned} \psi_{si} &= \frac{1}{h\pi}(\gamma_{si} - \gamma_{si}') + \int \Delta_{s\sigma} T_{\sigma i} d\sigma, \\ \psi_{se} &= \frac{-1}{h\pi}(\gamma_{se} + \gamma_{se}') + \int \Delta_{s\sigma} T_{\sigma e} d\sigma, \end{aligned}$$

where in both cases

$$(4)\qquad \gamma_{sp} = \frac{\partial}{\partial n_s} T_{sp}, \quad \gamma'_{sp} = \frac{1}{h\pi}\int \gamma_{\sigma p} \frac{\partial}{\partial n_s} T_{s\sigma} d\sigma.$$

A similar investigation is carried through for the second boundary problem; here the result is somewhat simpler. The author now proceeds to investigate the relation between the fundamental distributions on a surface and its transform by reciprocal radii, and furthermore gives a detailed discussion of the special cases of the circle, sphere, and ellipse. Certain convergence investigations concerning the Neumann-Robin methods conclude the memoir.

T. H. GRONWALL.

Physical and chemical Constants and some mathematical Functions. By G. W. C. KAYE, the National Physical Laboratory, England, and T. H. LABY, professor of physics, Wellington, N. Z. London, Longmans, Green, and Co., 1911. 153 pp.

THE authors have written this book to supply the need for a set of up-to-date English physical and chemical tables of convenient size and moderate price. In comparison with larger works such as those of Landolt, Bornstein, and Meyerhoffer this little book contains more information than its relative size indicates. In fact for class use where one illustration of a given type is good as another this book is ample. For the investigator desiring information of a very particular nature it may not be sufficient.

In most cases a table is accompanied by a brief statement containing definitions and standard formulas. Many references are given both to standard works and to observers for particular measurements. Interpolations and extrapolations are indicated.

The book contains several hundred tables covering subjects in general physics, astronomy, heat, sound, light, electricity, magnetism, radioactivity and gaseous ionization, and chemistry. At the end are tables of e^{-x}, four and five-place logarithms, reciprocals, squares, and natural trigonometric functions.

H. B. PHILLIPS.

NOTES.

THE annual list of American doctorates published in *Science* presents, for the academic year 1913–1914, 502 names, of which 241 are credited to the sciences. The following 27 successful candidates offered mathematics as major subject (the titles of the theses are appended): E. S. ALLEN, Harvard, "Su alcuni caraterri di una serie algebrica, e la formola di de Jonquières per serie qualsiasi;" R. D. BEETLE, Princeton, "Congruences associated with a one-parameter family of curves;" Miss S. R. BENEDICT, Michigan, "A comparative study of the early treatises introducing into Europe the Hindu art of reckoning;" B. A. BERNSTEIN, California, "A complete set of postulates for the logic of classes expressed in terms of the operation 'exception' and a proof of the independence of a set of postulates due to Del Rè;" J. A. BULLARD, Clark, "On the structure of finite continuous groups;" R. W. BURGESS, Cornell, "The uniform motion of a sphere through a viscous liquid;" Miss L. D. CUMMINGS, Bryn Mawr, "On a method of comparison for triple-systems;" J. W. GAIN, Johns Hopkins, "Linear combinants of ternary forms;" H. GALAJIKIAN, Princeton, "A type of non-linear integral equations;" H. C. GOSSARD, Johns Hopkins, "On a special elliptic ruled surface of the ninth order;" G. H. GRAVES, Columbia, "Complete linear systems of algebraic curves of least order of genera three and four;" L. M. KELLS, Columbia, "Complete characterization of dynamical trajectories in n-space;" E. A. T. KIRCHER, Illinois, "Group properties of the residue classes of certain Kronecker modular systems and some related generalizations in number theory;" W. C. KRATHWOHL, Chicago, "Modular invariants of two pairs of cogredient variables;" T. E. MASON, Indiana, "Character of the solutions of certain functional equations;" L. C. MATHEWSON, Illinois, "Theorems on the groups of isomorphisms of certain groups;"

Miss B. I. MILLER, Johns Hopkins, "A new canonical form of the elliptic integral;" L. W. MISER, Chicago, "On linear homogeneous differential equations with elliptic function coefficients;" F. M. MORRISON, Chicago, "On the relation between some important notions of projective and metrical differential geometry;" E. J. MOULTON, Chicago, "On figures of equilibrium of a rotating heterogeneous fluid body;" R. B. ROBBINS, Harvard, "The calculus of variations as the limit of a problem in minimizing an algebraic sum;" W. F. SHENTON, Johns Hopkins, "Linear combinants of systems of binary forms with the syzygies of the second degree connecting them;" S. P. SHUGERT, Pennsylvania, "König's resolvents and other types of symmetric functions;" J. M. STETSON, Princeton, "Conjugate systems both of whose Laplace transforms are lines of curvature;" Miss A. H. TAPPAN, Cornell, "Plane sextic curves invariant under birational transformations;" L. A. H. WARREN, Chicago, "A class of asymptotic orbits in the problem of three bodies;" Miss M. M. YOUNG, Johns Hopkins, "Dupin's cyclide as a self-dual surface."

THE course of lectures on General analysis to be given by Professor M. FRÉCHET at University of Illinois has been cancelled on account of the war.

THE Berlin Academy of Science has awarded Dr. P. KOEBE, of Leipzig, a prize of 5,000 marks for his work on the theory of functions.

THE Paris Academy of Sciences has granted 4,000 francs from the Bonaparte fund to Professor H. ANDOYER to assist in the publication of a set of trigonometric tables.

PROFESSOR KARL RUNGE, of the University of Göttingen, has been chosen Prorektor for the coming year.

IN honor of his holding the doctorate fifty years, the Technical School at Munich has conferred its honorary doctorate in engineering on Professor A. VON BRILL of Tübingen.

DR. A. N. WHITEHEAD of University College, University of London, has been appointed to the newly established chair of applied mathematics in the Imperial College of Science and Technology of London. Mr. G. N. WATSON, fellow of Trinity College, Cambridge, has been appointed to the staff of the University College in succession to Dr. Whitehead.

PROFESSOR G. T. ROGERS, of the New Mexico Normal School, has been elected assistant dean and professor of mathematics in the Normal University of New Mexico.

AT the University of California, Dr. C. E. BROOKS has been appointed assistant professor of mathematics and insurance, and Professor T. M. PUTNAM has been promoted to an associate professorship of mathematics.

PROFESSOR L. L. DINES, of the University of Arizona, has been appointed assistant professor of mathematics in the University of Saskatchewan.

AT the University of Kansas, Professor J. N. VAN DER VRIES has been promoted to a full professorship of mathematics. Dr. E. B. STOUFFER, of the University of Illinois, has been appointed assistant professor of mathematics, and Mr. E. H. CARUS instructor in mathematics. Professor Marion B. WHITE has resigned, to accept an appointment as associate professor of mathematics and dean of women at the Michigan state normal college at Ypsilanti.

AT the University of Michigan Dr. W. W. KÜSTERMANN and Mr. V. H. WELLS have been appointed instructors in mathematics.

DR. W. C. GRAUSTEIN, of Harvard University, has been appointed instructor in mathematics in the Rice Institute, Houston, Texas.

DR. F. B. WILEY has been appointed professor of mathematics in Denison University.

DR. J. L. JONES has been appointed instructor in mathematics in the University of Pittsburgh.

DR. M. G. GABA has been appointed instructor in mathematics in the Carnegie institute of technology at Pittsburgh.

Dr. J. M. Stetson has been appointed instructor in mathematics in the University of Alberta.

Professor W. H. Garrett, of Baker University, has received a year's leave of absence, which he will spend at Harvard University as student instructor.

Miss M. Daniels and Mr. H. Latimer have been appointed instructors in mathematics at the Iowa state college.

Miss M. A. Colpitts has been appointed instructor in mathematics at Tabor College.

Dr. W. J. Milne, president of the Albany normal college, died September 4 at the age of 71 years. Dr. Milne had been a member of the American Mathematical Society since 1906.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Antomari (X.). Cours de géométrie descriptive. Paris, Vuibert, 1914. 8vo. 641 pp. Relié. Fr. 12.50

Appell (P.). See Encyclopédie.

Arcais (F. di). Analisi infinitesimale. 2 volumes. Padova, 1913. L. 30.00

Bachmann (P.). See Klein (F.).

Backes (W.). Weitere Beweise für den Fermatschen Satz. 2te umgearbeitete und vermehrte Auflage von "Ein Beweis fur den Fermatschen Satz." Mainz, 1914. 8vo. 18 pp. M. 3.00

Baroni (E.). See Questioni.

Barthel (E.). Vertikaldimension und Weltraum. Leipzig, Hillmann, 1914. 28 pp. M. 0.75

Bernoulli (J.). Die erste Integralrechnung. Auswahl aus den mathematischen Vorlesungen über die Methode der Integrale und anderes (1691–92). Aus dem Lateinischen übersetzt und herausgegeben von G. Kowalewski. Berlin, 1914. 8vo. 8+285 pp. M. 5.00

Bôcher (M.). An introduction to the study of integral equations. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 10.) 2d edition. Cambridge, University Press, 1914. 8vo. 71 pp. 2s. 6d.

Borel (E.). Le hasard. Paris, Alcan, 1914. (Nouvelle collection scientifique; directeur, E. Borel.) 12mo. 4+312 pp. Fr. 3.50

——. See Niewenglowski (B.).

Boutroux (P.). See Volterra (V.).

BRAUDE (L.). Les coordonnées intrinsèques. Théorie et applications. (Collection Scientia, No. 34.) Paris, Gauthier-Villars, 1914. 8vo. 100 pp. Fr. 2.00

BRENDEL (M.). See KLEIN (F.).

BURALI-FORTI (C.) e MARCOLONGO (R.). Analyse vectorielle générale, Volume II. Applications à la mécanique et à la physique. Pavia, Mattei, 1913. 8vo. 12+143 pp. L. 5.00

DANIELE (—.). See QUESTIONI.

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome II, volume 5, fascicule 2: Fonctions sphériques, exposé d'après A. Wangerin par A. Lambert, avec une note de P. Appell et A. Lambert, pp. 161–230; Généralisations diverses des fonctions sphériques, exposé par P. Appell et A. Lambert, pp. 231–268. Paris, Gauthier-Villars, 1914. 8vo. Fr. 4.80

ENRIQUES (F.). Problems of science. Authorized translation by Katherine Royce, with an introduction by Josiah Royce. Chicago, Open Court, 1914. 8vo. 16+392 pp. Cloth. $2.50

——. See QUESTIONI.

FORSYTH (A. R.). Lectures introductory to the theory of functions of two complex variables. Cambridge, University Press, 1914. 8vo. 16+281 pp. 10s.

FORT (O.) und SCHLÖMILCH (O.). Lehrbuch der analytischen Geometrie. In 2 Teilen. II. Teil: Analytische Geometrie des Raumes. 7. Auflage, bearbeitet von R. Heger. Leipzig, Teubner, 1913. 8vo. 8+326 pp. Cloth. M. 6.80

GAZZANIGA (P. E.). Nuovo ed importante teorema di geometria e sue conseguenze: geometria nuova, memoria. Piacenza, Del-Maino, 1914. 8vo. 35 pp. L. 1.00

GIBSON (G. A.). Napier and the invention of logarithms. (*Tr. Royal Philosophical Society of Glasgow.*) Glasgow, Carter and Pratt, 1914. 8vo. 24 pp.

GOURSAT (E.). Cours d'analyse mathématique. Tome III, 1er et 2e fascicules. Paris, Gauthier-Villars, 1913–1914. 8vo. 544 pp. Complete volume to cost subscribers Fr. 20.00

HADAMARD (J.). See VOLTERRA (V.).

HALL (T. P.). A geometrical vector algebra. Vancouver, B. C., Western Specialty Co., 1914. 8vo. 2+30 pp.

HAUSDORFF (F.). Grundzüge der Mengenlehre. Leipzig, Veit, 1914. 8vo. 8+476 pp. M. 18.00

HEGER (R.). See FORT (O.).

HOBSON (E. W.). John Napier and the invention of logarithms. Cambridge, University Press, 1914. 8vo. 48 pp. Cloth. 1s. 6d.

KENNELLY (A. E.). Chart atlas of complex hyperbolic and circular functions. Cambridge, Mass., Harvard University Press, 1914. Folio. 23 charts. $4.00

KLEIN (F.) und BRENDEL (M.). Materialen fur eine wissenschaftliche Biographie von Gauss. Gesammelt von F. Klein und M. Brendel. Heft I: Ueber Gauss' zahlentheoretische Arbeiten, von P. Bachmann. Leipzig, Teubner, 1911. 8vo. 2+144 pp. M. 2.00

——. Materialen für eine wissenschaftliche Biographie von Gauss. Heft II: C. F. Gauss, Fragmente zur Theorie des arithmetisch-geometrischen Mittels aus den Jahren 1797–1799. Heft III: Ueber Gauss' Arbeiten zur Funktionentheorie, von L. Schlesinger. Leipzig, Teubner, 1912. 8vo. 2+144 pp. M. 4.40

KÖNIG (J.). Neue Grundlagen der Logik, Arithmetik und Mengenlehre. Leipzig, Veit, 1914. 8vo. 8+259 pp. M. 8.00

KOWALEWSKI (G.). See BERNOULLI (J.).

LAISANT (C. A.). Mathematics. New York, Doubleday, 1914. $0.50

LAMBERT (A.). See ENCYCLOPÉDIE.

LANGEVIN (P.). See VOLTERRA (V.).

LÉVY (P.). Sur les équations intégro-différentielles définissant des fonctions de lignes. (Thèse.) Paris, Gauthier-Villars, 1914. 4to. 4+122 pp. Fr. 6.00

LINDEMANN (F. und L.). See POINCARÉ (H.).

LOEBER (K.) Beiträge zur Losung und Geschichte des Malfattischen Problems und seinen Erweiterungen. (Diss. Halle.) Berlin, H Blankes Buchdruckerei, 1914. 8vo. 8+63 pp.+3 plates. M. 2 00

LONGLEY (W. R.). Tables and formulas for solving numerical problems in analytical geometry, calculus and applied mathematics. Boston, Ginn, 1913. 6+61 pp. $0.50

MAITLAND (F.). See POINCARÉ (H.).

MANNING (H. P.). Geometry of four dimensions. New York, Macmillan, 1914. 8vo. 10+348 pp.

MARCOLONGO (R.). See BURALI-FORTI (C).

NIEWENGLOWSKI (B.). Cours de géométrie analytique. 2e édition. Tome III: Géométrie dans l'espace (avec une note de E. Borel, sur les transformations en géométrie) Paris, Gauthier-Villars, 1914. 8vo. 4+608 pp. Fr. 14.00

PICARD (E.). L'oeuvre de Henri Poincaré. (*Annales scientifiques de l' Ecole Normale Supérieure.*) Paris, Gauthier-Villars, 1913. 4to. 22 pp. Fr. 2.00

POINCARÉ (H.). Wissenschaft und Hypothese. Autorisierte deutsche Ausgabe mit erlauternden Anmerkungen von F. und L. Lindemann. 3. Auflage. (Wissenschaft und Hypothese, Band L.) Leipzig, Teubner, 1914. 8vo. 16+346 pp. M 4.80

——. Science and method. Translated by Francis Maitland with an introduction by the Hon. Bertrand Russell. London, Nelson, 1914. 6s.

QUESTIONI riguardanti le matematiche elementari. Articoli di Baroni, Daniele, Vitali, ed altri, raccolte e coordinate da F. Enriques. Volume II: Problemi classici della geometria; numeri primi e analisi indeterminata; massimi e minimi. Bologna, Zanichelli, 1914. 8vo. 8+811 pp. L. 30.00

ROYCE (K. and J.). See ENRIQUES (F.).

RUSSELL (B.). See POINCARÉ (H.).

SALOMON (C.). Nouveaux essais de magie arithmétique polygonale. Etoiles magiques à 10 et 12 branches (30, 36, 48 points) et hexagones et octogones magiques. Paris, Gauthier-Villars, 1913. 8vo. 26 pp. Fr. 1.50

SCHLESINGER (L.). See KLEIN (F.).

SCHLÖMILCH (O.). See FORT (O.).

SMART (E. H.). First course in projective geometry. London, Macmillan, 1913. 24+273 pp. Cloth. 7s. 6d.

SOMMERVILLE (D. M. Y.). The elements of non-euclidean geometry. (Bell's Mathematical Series.) London, Bell, 1914. 8vo. 16+274 pp. Cloth. 5s.

VITALI (G.). See QUESTIONI.

VOLTERRA (V.). Henri Poincaré. L'oeuvre scientifique, l'oeuvre philosophique, par Vito Volterra, Jacques Hadamard, Paul Langevin, Pierre Boutroux. (*Revue du Mois.*) (Nouvelle collection scientifique; directeur, E. Borel.) Paris, Alcan, 1914. 16mo. Fr. 3.50

WANGERIN (A.). See ENCYCLOPÉDIE.

WATSON (G. N.). Complex integration and Cauchy's theorem. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 15.) Cambridge, University Press, 1914. 8vo. 79 pp. 3s.

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ON A GENERALIZATION OF A THEOREM OF DINI ON SEQUENCES OF CONTINUOUS FUNCTIONS.

BY DR. T. H. HILDEBRANDT.

(Read before the American Mathematical Society at Chicago, April 10, 1914.)

WE propose in this note to give a generalization of the following theorem of Dini*: "If a monotonic sequence of functions continuous on a closed interval converges to a continuous function, the convergence is uniform."

The double sequence analogue of this theorem proves to be of importance in our generalization. We embody it in the following

LEMMA. *If a double sequence* a_{mn} *is monotonic non-decreasing in* m *for every* n, *and if* $L_mL_na_{mn} = L_nL_ma_{mn}$, *all the limits being supposed to exist, then* L_ma_{mn} *and* L_na_{mn} *converge uniformly and the double limit* $L_{mn}a_{mn}$ *exists and is equal to the iterated limits.*

The proof of the uniformity of convergence of L_ma_{mn} is part of Theorem I of the paper by the author in the *Annals of Mathematics*, series 2, volume 14, page 81. This uniformity has as a direct consequence the existence of the double limit equal to the iterated limits, which in turn implies the uniformity of L_na_{mn}.

For the purposes of generalization, consider a class $\mathfrak{D}$ of elements unconditioned excepting for the existence within the class of some definition of limit, i. e., some means of determining whether a sequence of elements has a limit and what this limit is.† Then it is possible to define the concepts *limiting element, closed and compact* relative to subclasses $\mathfrak{R}$ of $\mathfrak{D}$.‡ Also if μ is a real-valued function on $\mathfrak{R}$, we can define the notion of *continuity*, as well as *equal continuity*, as applied to a set of functions. In such a situation we are able to state the following

THEOREM. *If* $\mathfrak{R}$ *is a closed and compact subclass of* $\mathfrak{D}$, *and* μ_{nr} *is a monotonic sequence of functions continuous on* $\mathfrak{R}$ *and*

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* Cf. Dini-Lüroth-Schepp: Theorie der Funktionen, p. 148.

† Cf. *Amer. Journ. of Mathematics*, vol. 34, p. 241.

‡ Cf. Fréchet: "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo*, vol. 22 (1906), pp. 6, 7, 11.

converging to the continuous function μ_r, *then the convergence is uniform and the functions* μ_{nr} *are equally continuous.*

We prove the uniformity of convergence of μ_{nr}. Since $L_n\mu_{nr} = \mu_r$ for every r of the class $\mathfrak{R}$, we have: for every positive e there exists an n_{er} such that if $n \geqq n_{er}$ we have $|\mu_{nr} - \mu_r| \leqq e$. Suppose that we have selected for each e and r the smallest possible value as n_{er}. Then we wish to show that for each e, n_{er} is bounded on the class $\mathfrak{R}$. Suppose, if possible, this were not so for some particular e. Then for every n, there would exist an r_n, such that $n_{er_n} > n$, i. e., $|\mu_{nr_n} - \mu_{r_n}| > e$. On account of the convergence of μ_{nr} to μ_r for every r, no element can recur infinitely often in the set r_n. Then since the class $\mathfrak{R}$ is compact and closed, there will exist a subsequence $r_{n_m} = r'_m$ of r_n, and an r, such that $L_m r'_m = r$. Consider the double sequence $\mu_{nr'_m}$. It is monotonic non-decreasing in n for every m. Moreover on account of the continuity of μ_{nr} and μ_r, we have $L_m L_n \mu_{nr'_m} = L_n L_m \mu_{nr'_m}$. It therefore fulfils the conditions of our lemma, and it follows that $L_n\mu_{nr'_m}$ converges uniformly; i. e., for every positive e there will exist an n_e, independent of m, such that $|\mu_{nr'_m} - \mu_{r'_m}| \leqq e$. By taking the e as the one presupposed above, and $n > n_m$, we obtain a contradiction.

The equal continuity of the functions μ_{nr} is a direct consequence of their uniformity of convergence and continuity.

To obtain a further generalization we presuppose another general class $\mathfrak{P}$. In $\mathfrak{P}$ we shall suppose that there is defined an order relation between triplets of elements: $B_{p_1p_2p_3}$, comparable to $p_1 \leqq p_2 \leqq p_3$. We shall suppose that there exists in the class also the concept of limit, subject to the condition that, if $L_n p_n = p$, then there exists a subsequence having the same limit, such that $B_{p_{n_m}p_{n_{m+1}}p}$ for every m, or $B_{pp_{n_{m+1}}p_{n_m}}$ for every m. If μ is a real valued function on $\mathfrak{S}$, a subclass of $\mathfrak{P}$, then μ is said to be *monotonic non-decreasing* on $\mathfrak{S}$ if for every triplet s_1, s_2, s_3, of $\mathfrak{S}$ such that $B_{s_1s_2s_3}$ we have $\mu_{s_1} \leqq \mu_{s_2} \leqq \mu_{s_3}$. Finally in the composite class $\mathfrak{P}\mathfrak{Q}$,* we obtain a double limit, viz., $L_{mn}p_mq_n = pq$ is equivalent to $L_mp_m = p$ and $L_nq_n = q$. This enables us to define a continuity of functions on a composite range, similar to that of continuity of functions of two variables, viz., μ is continuous on $\mathfrak{S}\mathfrak{R}$ if $L_{mn}s_mr_n = sr$ implies $L_{mn}\mu_{s_mr_n} = \mu_{sr}$. Then we have the following

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* Cf. Moore: Introduction to General Analysis, p. 90.

THEOREM. *If $\mathfrak{S}$ and $\mathfrak{R}$ are closed and compact subclasses of $\mathfrak{P}$ and $\mathfrak{Q}$, respectively, if further μ_{sr} is continuous on $\mathfrak{S}$ for every r and on $\mathfrak{R}$ for every s, and if moreover μ_{sr} is monotonic non-decreasing on $\mathfrak{S}$ for every r, then μ_{sr}, considered as a set of functions on $\mathfrak{R}$, are equally continuous, as well as μ_{sr} considered as a set of functions on $\mathfrak{S}$, and μ_{sr} is continuous on $\mathfrak{S}\mathfrak{R}$.*

The proof that μ_{sr}, considered as a set of functions on $\mathfrak{R}$, are equally continuous is an indirect one. The assumption that μ_{sr} is not equally continuous on $\mathfrak{R}$ is shown to be untenable by a use of the property of limit in terms of B, the monotoneity and continuity of μ_{sr}, and the preceding theorem. The details are easily supplied. The equal continuity of the set μ_{sr} on $\mathfrak{S}$, and the continuity on $\mathfrak{S}\mathfrak{R}$ follow at once from the equal continuity on $\mathfrak{R}$.

By specializing the classes $\mathfrak{P}$ and $\mathfrak{Q}$, we get some interesting theorems in special fields. If we take $\mathfrak{P} = 1, 2, 3, \cdots, \infty$, with $B_{p_1p_2p_3}$ defined as $p_1 \leqq p_2 \leqq p_3$, and $\mathfrak{Q}$ as the interval $0 \leqq x \leqq 1$, and note that equal continuity on $1, 2, 3, \cdots, \infty$ is uniform convergence, we get the Dini theorem stated at the outset. If $\mathfrak{P}$ is the linear interval $0 \leqq x \leqq 1$, and $B_{p_1p_2p_3}$ is the same as $p_1 \leqq p_2 \leqq p_3$, and $\mathfrak{Q}$ is the set $1, 2, 3, \cdots, \infty$, we have:

*If a sequence of monotonic non-decreasing functions continuous on a closed interval converges to a continuous function, the convergence is uniform, and the set of functions are equally continuous.**

If $\mathfrak{P}$ is the linear interval $0 \leqq x \leqq 1$, with $B_{p_1p_2p_3}$ equivalent to $p_1 \leqq p_2 \leqq p_3$ and $\mathfrak{Q}$ is the linear interval $0 \leqq y \leqq 1$, we have:

If $f(x, y)$, defined for $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, is continuous in x for every y, and in y for every x, and is also monotonic non-decreasing in x for every y, then $f(x, y)$ is continuous in x and y simultaneously.

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* Cf. Buchanan and Hildebrandt: *Annals of Mathematics*, ser. 2, vol. 9, p. 123. It is interesting to observe that this theorem and the Dini theorem are special cases of the same theorem.

NOTE ON REMOVABLE SINGULARITIES.

BY MR. W. E. MILNE.

IN his dissertation[*] Kistler proves the following theorem:

"*Let* $f(z_1, \cdots, z_n)$ *be analytic throughout the neighborhood of a point* $(a_1, \cdots, a_n)$ *with the exception at most of the points of a finite number of analytic manifolds, each of which is at most* $(2n - 4)$*-dimensional. Then it is analytic in the excepted points also, if properly defined there.*"

A similar theorem is given below. The two theorems differ in two respects. (1) Kistler's theorem requires the excepted points to lie on analytic manifolds, while the present one does not; (2) the present theorem requires that for *every*[†] pair of variables the excepted locus reduce to isolated points when the remaining $n - 2$ variables are fixed, while Kistler's theorem requires this to hold simply for *one* pair. But it is to be noted that the hypotheses of the present theorem will in general[‡] be fulfilled if the singular manifolds are analytic.

The theorem is as follows:

THEOREM. *Let the function* $\varphi(z_1, \cdots, z_n)$ *of* n *complex variables,* $n > 2$, *be analytic in the region* $(S_1, \cdots, S_n)$ *except for the points of a* $(2n - 4)$*-dimensional locus of such character that when any* $n - 2$ *of the variables are given any fixed values in their respective regions, and* z_i *and* z_j *alone vary, then the singularities occur only at isolated points* (a_i, a_j) *in* (S_i, S_j). *Under these conditions* φ *has a limit in the points of the singular locus, and if defined as equal to its limit, will be analytic without exception in* $(S_1, \cdots, S_n)$.

Proof: Let $z_3, \cdots, z_n$ be held fast at $(a_3, \cdots, a_n)$ any point of $(S_3, \cdots, S_n)$, and consider φ as a function of z_1 and z_2. From the hypothesis we see that singularities can occur only at isolated points of (S_1, S_2). But isolated singularities for a function of two complex variables are removable; so φ, if properly defined, will be analytic in z_1 and z_2 throughout (S_1, S_2).

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[*] "Ueber Funktionen von mehreren komplexen Veranderlichen," Basel, 1905. § 7.

[†] Evidently all that is really necessary is that with each variable z_i it be possible to associate another z_j such that if the remaining $n - 2$ are fixed, the singular points in z_i, z_j are isolated.

[‡] I hope shortly to be able to show that the excepted cases can always be avoided by a suitable linear transformation of the independent variables, and hence that the words "in general" can be replaced by "always."

Give to z_2 any value a_2 in S_2, and φ will be analytic in z_1 alone. This holds for every choice of the fixed values assigned to $z_2, \cdots, z_n$. In a similar manner we find φ analytic in each remaining variable alone.

Now apply the theorem of Hartogs[*] which states that if a function of n complex variables is analytic in each one separately, it is analytic in all n variables taken together. Hence φ is analytic throughout $(S_1, \cdots, S_n)$.

HARVARD UNIVERSITY,
May, 1914.

CONCERNING A CERTAIN TOTALLY DISCONTINUOUS FUNCTION.

BY PROFESSOR K. P. WILLIAMS.

(Read before the American Mathematical Society, October 31, 1914.)

ONE of the most important properties of a continuous function is that it actually assumes every value between any two of its values. It is well known that a function can, however, possess this property without being continuous. An actual example to illustrate this seems to have been first given by Darboux in 1875. A function that is sometimes cited in this connection is due to Mansion.[†] The function that the latter gives actually takes all values between any two, but is discontinuous at the single point $x = 0$. Functions of this sort can be easily constructed by arbitrarily assigning the values at certain points, according to the function concept of Dirichlet. More interest would therefore attach to such a function if it is given by one and the same expression throughout its region of definition. The function given by Mansion does not, however, possess this property; for it contains the function $E(x)$, defined, as in number theory, as the integer equal to, or next smaller than x.

The purpose of this note is to give a function that takes every value between 0 and 1 inclusive, when x varies over the closed interval (0, 1), but which is discontinuous at every point. This function will, furthermore, be represented by one and the same analytical expression throughout its whole region of definition.

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[*] *Math. Ann.*, vol. 62 (1905), p. 1.

[†] "Continuité au sens analytique et continuité au sens vulgaire," in *Mathesis*, 1899.

Let $f(x)$ be equal to zero at the rational points of the interval $(0, 1)$, and equal to 1 at the irrational points. We first obtain for this function an expression which is a modification of the one given by Hankel in his celebrated memoir on oscillating and discontinuous functions.*

Let

$$\varphi(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi x}{2n+1};$$

then, as is well known,

$$\varphi(x) = \quad 1, \text{ for } \quad 0 < x < 1,$$

$$\varphi(x) = -1, \text{ for } -1 < x < 0,$$

while

$$\varphi(0) = \varphi(\pm 1) = 0.$$

This gives us for $f(x)$ the following expression:

$$f(x) = \prod_{n=1}^{\infty}\{\varphi(\sin n\pi x)\}^2.$$

The expression which Hankel gives for $f(x)$ defines it, in reality, only for the irrational points; so that its values at the rational points must be assigned.†

We next define the function $F_1(x)$ by the relation

$$F_1(x) = x + (1 - 2x)f(x).$$

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* *Math. Annalen*, vol. 20, pp. 63–112.

† Hankel puts

$$f(x) = \left(\sum_{n=1}^{\infty}\frac{1}{n^\mu}\right)\Big/ g(x),$$

where $\mu > 1$, and

$$g(x) = \sum_{n=1}^{\infty} 1/n^\mu[\varphi(\sin n\pi x)]^2.$$

He says that at the rational points $g(x)$ becomes infinite; thus making $f(x)$ equal to zero at those points. While it is true that at the rational points the denominators of some of the terms in $g(x)$ become zero, those terms do not behave in any sense as a function does at a pole. The terms abruptly take the form 1/0, and as this symbol is undefined, we cannot regard the series as defining $g(x)$ at the rational points. Other writers have given the function as Hankel gave it.

As Pringsheim has shown, we also have

$$f(x) = 1 - \lim_{m=\infty}\left[\lim_{n=\infty}(\cos m!\pi x)^{2n}\right].$$

Consequently we have

$$F_1(x) = 1 - x, \text{ for } x \text{ irrational and } 0 \leqq x \leqq 1,$$

$$F_1(x) = x, \text{ for } x \text{ rational and } 0 \leqq x \leqq 1.$$

The function $F_1(x)$ accordingly takes all values between 0 and 1 inclusive when x varies over the closed interval (0, 1). It is, in addition, discontinuous at every point, save the point $x = \frac{1}{2}$. We next modify the function so that $x = \frac{1}{2}$ is also a point of discontinuity.

Let

$$\bar{\varphi}(x) = \varphi(2x);$$

then, from the above values of $\varphi(x)$, and the fact that it is periodic, we obtain

$$\bar{\varphi}(0) = \bar{\varphi}(\tfrac{1}{2}) = \bar{\varphi}(1) = 0;$$

$$\bar{\varphi}(x) = 1, \text{ for } 0 < x < \tfrac{1}{2};\ \bar{\varphi}(x) = -1, \text{ for } \tfrac{1}{2} < x < 1.$$

Consider now the function

$$F_2(x) = \frac{(1-x)4^x}{2}[1 - \bar{\varphi}^2(x)] \cos 2\pi x,$$

where 4^x denotes the arithmetic root.

From the above table of values of $\varphi(x)$ we have at once

$$F_2(0) = \tfrac{1}{2};\ F_2(x) = 0,\ 0 < x < \tfrac{1}{2};\ F_2(\tfrac{1}{2}) = -\tfrac{1}{2},$$
$$F_2(x) = 0,\ \tfrac{1}{2} < x \leqq 1.$$

We construct finally the function

$$F(x) = F_1(x) + F_2(x).$$

It is apparent that $F(x)$ is obtained from $F_1(x)$ by merely interchanging the values at the two points $x = 0$ and $x = \frac{1}{2}$. From the properties of $F_1(x)$ it then follows that $F(x)$ takes all values between 0 and 1 inclusive, and is, furthermore, discontinuous at every point. We see, finally, that $F(x)$ can be represented by a single analytical expression throughout the interval (0, 1); for we have expressions for all the functions contained in it. We consequently have in $F(x)$ a function which possesses all the properties desired.

We shall note a few additional properties of the function we have obtained.

In addition to being single valued, $F(x)$ assumes a given value but once. We can thus regard it as giving a one-to-one transformation of the interval (0, 1) into itself, which is everywhere discontinuous. At every point save $x = \frac{1}{2}$ the function has no limit; that is, every point, except $x = \frac{1}{2}$, is a point of discontinuity of the second kind. It is also apparent that both the greatest and least values approached at a point are continuous functions.

INDIANA UNIVERSITY,
May, 1914.

PROOF OF THE CONVERGENCE OF POISSON'S INTEGRAL FOR NON-ABSOLUTELY INTEGRABLE FUNCTIONS.

BY DR. W. W. KÜSTERMANN.

IN the following pages I propose to give a proof of the

THEOREM: *If* $f(x)$ *is a real, periodic function, of period* 2π, *which in the interval* $(0, 2\pi)$ *has a proper or improper integral in the sense of Lebesgue, Harnack-Riemann, or Harnack-Lebesgue-Hobson,** *then*

$$\lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\alpha) \frac{1 - r^2}{1 + r^2 - 2r \cos (\alpha - x)} d\alpha = \lim_{t \to 0} \tfrac{1}{2}[f(x + t) + f(x - t)]$$

at every point x *where the limit on the right hand side exists.*

This theorem includes in particular the case where $f(x)$ remains finite—disposed of by Schwarz,† and the case where $f(x)$ becomes infinite at an infinite number of points, but has an *absolutely* convergent improper integral—discussed by Hobson and others.‡ Moreover, it goes farther, in that it also covers the case where $f(x)$ becomes infinite in such a way as to be *not absolutely* integrable in the vicinity of a point, or, for that matter, of a non-dense, closed set of points of content zero. All former proofs, as far as I know, fail to cover this case, since they require at some place or other that $|\int f(\alpha)d\alpha|$ be replaced by $\int|f(\alpha)d\alpha|$.* The subsequent proof rests upon an application of the second mean value theorem, established by Hobson† in its most general form. For the present purpose Hobson's results may be stated as follows:

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* For these definitions see Hobson, Theory of Functions of a Real Variable, Cambridge, 1907.

† Schwarz, Math. Abhd., vol. 2, pp. 144 and 175.

‡ Most completely by Hobson, Theory of Functions of a Real Variable, p. 719; cf. also Bôcher, *Ann. of Math.*, 2d ser., vol. 7, p. 81; Fatou, *Acta Math.*, vol. 30, p. 335; Picard, Traité d'Analyse, 2d ed., vol. 1, p. 268; Forsyth, Theory of Functions., 2d ed., p. 450.

"If $\varphi(x)$ be a limited and monotonically decreasing positive function of the real variable x, and if $f(x)$ have an improper integral in (a, b), either absolutely or non-absolutely convergent in accordance with the definitions of Lebesgue, Harnack-Riemann, or Harnack-Lebesgue-Hobson, then

(I) $\varphi(x)f(x)$ has also an improper integral in (a, b) in accordance with the same definition;

(II) $\int_a^b \varphi(x)f(x)dx = A\int_a^X f(x)dx$, where A is any number such that $A \geqq \varphi(a+0)$ and X is a number in the interval (a, b)."

Now consider

$$\varphi_r(\alpha) = \frac{1-r^2}{1+r^2-2r\cos\alpha},$$

where $0 < r < 1$ in the interval $0 \leqq \alpha \leqq \pi$. The $\cos\alpha$ decreases monotonically from $+1$ to -1. The expression $1+r^2-2r\cos\alpha$, however, increases from $(1-r)^2$ to $(1+r)^2$ and remains always greater than zero. It follows that, for each $0 < r < 1$, $\varphi_r(\alpha)$ is a positive, monotonically decreasing function and satisfies the premises of the mean value theorem. If then $f(x)$ does likewise, we may conclude from (I) that Poisson's integral

$$P(x, r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\frac{1-r^2}{1+r^2-2r\cos(\alpha-x)}d\alpha$$

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[*] Thus the presence of just one singularity such as $\frac{\sin 1/\alpha}{\alpha}$ has at $\alpha = 0$ would question the existence of Poisson's integral and its limit for any point x whatever.

[†] Hobson, *Proc. Lond. Math. Soc.*, 2d ser., vol. 7 (1909), p. 14.

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\alpha)\,\frac{1-r^2}{1+r^2-2r\cos\alpha}\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\alpha)\varphi_r(\alpha)d\alpha$$

exists and has meaning. Now

$$\int \frac{(1-r^2)d\alpha}{1+r^2-2r\cos\alpha} = \arctan\left[\frac{1+r}{1-r}\tan\frac{\alpha}{2}\right],$$

hence

$$\int_{-\pi}^{\pi}\varphi_r(\alpha)d\alpha = 2\pi,$$

independent of r. Let

$$\lim_{t\to 0}\tfrac{1}{2}[f(x+t)+f(x-t)] = L, \tag{1}$$

then

$$2\pi\{P(x,r)-L\} = \int_{-\pi}^{\pi}[f(x+\alpha)-L]\varphi_r(\alpha)d\alpha$$

$$= 2\int_0^{\pi}\left[\frac{f(x+\alpha)+f(x-\alpha)}{2}-L\right]\varphi_r(\alpha)d\alpha,$$

since $\varphi_r(\alpha) = \varphi_r(-\alpha)$,

$$= 2\left\{\int_0^{\delta}+\int_{\delta}^{\pi}\right\}\left[\frac{f(x+\alpha)+f(x-\alpha)}{2} - L\right]\varphi_r(\alpha)d\alpha.$$

An arbitrarily small $\epsilon > 0$ having been prescribed, let δ be chosen in such a way that

$$\left|\frac{f(x+\alpha)+f(x-\alpha)}{2}-L\right| < \epsilon, \text{ for } 0 < \alpha < \delta.$$

That this is always possible follows from (1). Moreover, $\varphi_r(\alpha)$ being positive, $|\varphi_r(\alpha)| = \varphi_r(\alpha)$. Hence, applying the second mean value theorem (II) to $\int_{\delta}^{\pi}$, we have

$$\pi \left| P(x, r) - L \right| \leqq \epsilon \int_0^\delta \varphi_r(\alpha) d\alpha$$

$$+ \varphi_r(\delta) \cdot \left| \int_\delta^{X_r} \left[\frac{f(x+\alpha) + f(x-\alpha)}{2} - L \right] d\alpha \right|,$$

where $\delta \leqq X_r \leqq \pi$,

$$< \epsilon\pi + \varphi_r(\delta) G,^*$$

for $0 < r < 1$,

$$< \epsilon\pi + \frac{(1 - r^2)K}{\sin^2 \frac{1}{2}\delta}$$

where G and K are positive constants,

$$< \epsilon',$$

for r close enough to unity. Hence $\lim\limits_{r \to 1} P(x, r) = L$.

W. W. KÜSTERMANN.

ANN ARBOR,
May 2, 1914.

THE NAPIER TERCENTENARY CELEBRATION.

THE Napier Tercentenary Celebration was held at Edinburgh on July 24–28, 1914, with over three hundred visitors present. The ceremonies opened at the University of Edinburgh under the presidency of the Lord Provost of the city, the address of the day being delivered by Lord Moulton and relating to the probable reasons in Napier's mind for deciding upon a table of logarithms of sines and to his probable methods of computing. Brief addresses were given by four of the official delegates, Professors Andoyer and D'Ocagne of Paris, Smith of New York, and Bauschinger of Strassburg.

On Saturday morning the first session was held under the presidency of Professor Hobson of Cambridge. The following papers were presented:

DR. J. W. L. GLAISHER, of Cambridge: "The work of Napier."

Dr. Glaisher called attention to the paucity of notations

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[*] The argument here is based on the fact that a definite integral is a continuous function of its upper limit and as such has a finite maximum and minimum. Cf. E. H. Moore, *Transactions*, vol. 2, pp. 296 and 459.

and general symbols three hundred years ago to aid in the forming of clear conceptions of the nature of logarithms, and stated that an examination of Napier's work impressed one with his greatness as a genuine mathematical thinker rather than as the discoverer of an isolated detail.

DR. G. VACCA, of Rome: "The first Napierian logarithm before Napier."

In the absence of the author of the paper, Dr. Knott gave a brief summary. Dr. Vacca believes that a certain passage in Pacioli (1494), in relation to the computation of interest, has in it the essence of a logarithm. The rule given by Pacioli is substantially this: Divide 72 by the rate of interest and the quotient is the required number of years that it takes a sum to double itself at compound interest. Dr. Vacca shows that this number 72 is approximately 100 log 2 in Napier's system.

PROFESSOR G. A. GIBSON, of Glasgow: "On the transition from Napier's to Briggs's logarithms."

Professor Gibson showed that the improvement suggested by Briggs in his talk with Napier was not what we commonly look upon as the Briggs logarithms. He asserted that it was Napier himself who first suggested making log 1 equal to zero.

PROFESSOR D. E. SMITH, of New York: "Laws of exponents in the works of the sixteenth century."

In this paper was discussed the gradual rise of the laws of exponents in the sixteenth century, the statements being based upon extracts from the original works of various writers. It was the author's purpose to show that the fundamental laws of logarithms were well known as applied to integral exponents, and that Bürgi was led by them to prepare his Progress Tabulen (1620). The chief purpose, however, was to bring together for the use of students the actual words of the leading sixteenth century writers relating to the subject.

LIEUTENANT SALIH MOURAD, of Constantinople: "On the introduction of logarithms into Turkey."

Lieutenant Mourad showed that Ismail Effendi (Caliph Zade) introduced logarithms into Turkey exactly one hundred years after Napier's first publication of 1614.

PROFESSOR F. CAJORI, of Colorado Springs: "Algebra in Napier's day, and alleged prior inventions of logarithms."

The paper considered the claims presented by certain writers that Regiomontanus, Wright, and Bürgi, or some one of them, had anticipated Napier in the invention of logarithms. Professor Cajori weighed the evidence in a judicial manner and showed that there was nothing to substantiate a claim to priority on the part of any of these writers.

DR. D. M. Y. SOMMERVILLE, of St. Andrews: "On Napier's rules and trigonometrically equivalent polygons, with extension to non-euclidean space."

The second meeting was held on Monday, July 27, under the presidency of Professor D. E. Smith of New York, Dr. J. W. L. Glaisher of Cambridge, and Major P. A. MacMahon of London, who served during three sessions in the order named. The following papers were read:

PROFESSOR H. ANDOYER, of Paris: "On the history and method of construction of tables."

Professor Andoyer explained the construction of his recent (1911) tables which contain the logarithms of the trigonometric functions to seventeen significant figures for every hundredth of the quadrant, and to fourteen figures for every ten seconds. He also mentioned a work upon which he is now engaged, the calculation of a table of natural sines to fifteen significant figures, a work which he expects to finish in about two years.

PROFESSOR M. D'OCAGNE: "Communications relating to the history of calculating machines and to the development of nomography."

He stated that the principle of the Millionaire calculating machine was the invention (1893) of a young French mechanician, Léon Bollée, and that the earliest systematic application of nomograms is found in Margett's Longitude Tables of 1791.

MRS. EMMA GIFFORD, of Chard, Somerset: "On a recent table of sines."

Mrs. Gifford described her method of computing and checking her recent table of natural sines, which gives the sine for every second of the quadrant.

DR. J. R. MILNE, of Edinburgh: "On the arrangement of tables."

This paper was illustrated by lantern slides, and the questions of type, workmanship, paper, and general arrangement were discussed.

MR. H. S. GAY, of Shamokin, Pa.: "On a convenient formula for determining the angle, given the sine or cosine."

Mr. Gay presented a formula which easily gives an angle to a sufficiently close degree of approximation for practical purposes, when the cosine or sine is known.

MR. J. C. FERGUSSON, of Birmingham: "On the percentage unit of measuring angles."

The essential feature is the use of the tangent instead of the angle, and of the octant instead of the quadrant.

MR. W. SCHOOLING, of London: "On the calculation of logarithms."

The speaker showed how the logarithms of numbers, the logarithms of their reciprocals, and the gaussian logarithms, can be calculated by pure addition.

DR. A. HUTCHINSON, of Cambridge: "On graphic methods and on the use of the slide rule in crystallography."

DR. W. F. SHEPPARD, of Sutton: "On the constructing of tables."

The speaker described an extension of tables by an improved method of differences.

A few other papers were, in the absence of the authors, read by title.

The social functions were well attended by all who were attracted to Edinburgh by the celebration. On Friday evening a reception was given by the Lord Provost of Edinburgh and Council, followed by a concert. On Saturday afternoon a garden party was held at Merchiston Castle, and visitors were permitted to see the room in which it is said that Napier worked out his system of logarithms. In the evening there was an informal social gathering in the Students' Union at the University. On Monday afternoon the Royal Society of Edinburgh gave a farewell reception at the Society rooms.

On Sunday a special memorial service was held at St. Giles' Cathedral, the sermon being preached by the Rev. Dr. Fisher, of St. Cuthbert's, the church in which Napier worshipped and in the churchyard of which he is buried. Dr. Fisher spoke of Napier as a citizen and as a writer on questions of theology. The service was attended by a large number of those present at the celebration.

Mention should be made of the exhibit of Napieriana and of tables and calculating machines. Early editions of works on logarithms, Napier portraits, Napier rods ("bones"), and the priceless Dantzig copy of Bürgi's Progress Tabulen were displayed in cases. All the leading types of calculating machines, models of solids and surfaces, linkages, tide predicting machinery, and other material relating to computation and to those parts of geometry in which Napier had an interest, attracted much attention.

The celebration was followed by a colloquium held under the auspices of the Edinburgh Mathematical Society. Lectures were given by Professors D'Ocagne and Whittaker, and by Messrs. E. Cunningham and H. W. Richmond.

The papers will appear in a volume under the editorship of Dr. C. G. Knott. A description of the exhibit appeared in dignified book form under the direction of Mr. E. M. Horsburgh, and a copy was presented to each participant in the celebration.

Those who attended the celebration were unanimous in their expression of pleasure at the success of the meetings and at the hospitality shown by the citizens of Edinburgh, the Royal Society, the Mathematical Society, and the University.

DAVID EUGENE SMITH.

AN APPEAL TO PRODUCING MATHEMATICIANS.

BY GEORGE PAASWELL, C.E.

A PERUSAL of modern mathematical treatises and dissertations for degrees makes the lay mathematician (I should say rather the amateur mathematician) despair of ever keeping up with the modern trend of mathematical thought. The tone of modern works is not that of disseminating new ideas, but rather that of clothing ideas already familiar to readers in slightly different form; a tone surely not that of a text-book and therefore precluding from its readers those not familiar with the subject.

There is, however, a more serious arraignment to be made. Hardly any treatise has attempted to discuss or analyze the serious problems of the applied science professions. The profession of civil engineering is teeming with problems awaiting the solutions of a St. Venant, a Laplace, a Newton. The problems are vital; fundamental truths await discovery and elucidation. Innumerable functions await an Abel or Jacobi to bring them into existence.

Glance through the standard texts on applied mechanics and marvel at the appalling gaps in analysis the engineer must bridge with assumptions far from rigorous or satisfactory, to obtain any practical result.

The profession of building is the oldest in the world and yet how little we have progressed in the analysis of the stresses in the frame itself. The knowledge of stresses in a column remains practically in the state in which Euler left it. There are, indeed, some empirical column formulas, but an empirical formula is always a sad apology for lack of scientific knowledge. It may be said that empirical formulas have sufficed in the past for design, but the failures of structures designed on that basis most effectively answer that. The failure of the largest bridge in the world—the Quebec bridge—was due to lack of knowledge of the action of large compression members (columns) and the only path open to engineers was that of experimentation on larger size test pieces: mathematicians had failed them.

When the simple case of the analysis of flexure is attempted, there is either the faulty Bernoulli-Euler theory or else one must wade through appalling intricacies of analysis to gain the desired kernel of practical value which is then found to be limited in its application. When we come to the analysis of structures proper, there are either empirical formulas or faulty analysis. So far, I have had in mind homogeneous bodies; but when heterogeneous bodies are included, such as concrete, plain or reenforced, the ignorance of the action of the material itself adds to the sum total of our ignorance.

When we come to the analysis of statically indeterminate structures—a most fruitful field—there are several fundamental principles to be had for stress analysis; but when the structures become less simple, the intricacies and the tediousness of analysis become inhibitive to further discussion. Yet these are the types of structures that make for economy of design and for rigidity and permanence. Here are abundant fields for mathematical thought, and every thesis and article would bring us nearer to a solution of these problems.

The subject of moving bodies with their resultant impacts and vibrations is at present hopeless—even the empirical formulas fail us here—and we must make up for lack of knowledge by excessive and wasteful strength. Likewise the study of earth pressures both vertical and lateral exists in a series of empiricisms.

There is sore need of a text on the theory of elasticity which, while more or less free of the involved intricacies, shall give a firm basis for stress and strain analysis.

In this connection the curriculums of the applied science schools are at fault. Just sufficient mathematical knowledge is disseminated to enable the student to interpret a few simple formulas and theories. No attempt is made to inculcate a rigorous and fundamental knowledge of mathematics nor to give even a hint of the vast fields of advanced mathematical thought.

The study of mathematics, per se, is a most soul satisfying and alluring pleasure, and those to whom is granted the opportunity of devoting the greater part of their time to its perusal are most sincerely to be envied; yet society asks in return a solution of its most pressing problems.

It may be visionary dreams of the wildest type, but the day may come when, from the knowledge of the chemical constituents of a body, its crystallographic history and its temperature and space elements, its stress and strain relations shall at once be known and the effects of external loadings, static or otherwise, at once interpreted.

There is hardly a branch of mathematical analysis that could not add its quota to applied science: the calculus of variations and the potential function to the principles of least work and infinitesimal distortions; the elliptic and related functions to the study of the elastica and allied problems; and general analysis itself to the reduction of the resulting expressions of stress analysis.

I have enumerated but a few of the vast problems frequently confronting engineers in daily life and fervently hope that they may elicit some dissertations that may bring light on these vital problems.

SHORTER NOTICES.

Die Mathematik im Altertum und Mittelalter. By H. G. ZEUTHEN. Erster Abschnitt: *Entstehung und Entwicklung der Zahlen und des Rechnens.* B. G. Teubner, Leipzig und Berlin, 1912. 95 pages. Price unbound, 3 Marks.

THIS little work is a portion of Teil III of the volume entitled Die mathematischen Wissenschaften, in the work edited by Professor Hinnenberg, Die Kultur der Gegenwart. Teil III has for its title Die mathematischen und naturwissenschaftlichen Kulturgebiete, and the part relating to mathematics is under the editorial supervision of Professor Klein.

No opportunity was afforded, in the brief space allotted to Professor Zeuthen, for any new contribution to the history of mathematics. Therefore all that can be expected is a mere résumé of the leading contributions to the science in ancient and medieval times. Professor Zeuthen first treats of early arithmetic, beginning with the primitive number systems, passing to the early mechanical methods of computation, setting forth the difficulties of notation in ancient times, tracing rapidly the development of our numerals, and making clear the obstacles met by all early peoples in the handling of fractions. He then considers the applications of number to commerce, astronomy, mysticism, and puzzle problems, showing the relation of this work to the primitive algebra of the Egyptians.

He next takes up the geometry of the Egyptians and Greeks. While giving only a cursory glance at the development of the science, he takes occasion to refer to the fact that the Pythagorean triangle is mentioned in the Sulva-sutras, which he puts as late as the 5th or 4th century B.C. He does not venture, however, upon the question of the antiquity of the theorem in China, a problem which probably can be solved only by the rise of native scholars who can give us a careful textual criticism of the classics of their country. It is interesting to see that Professor Zeuthen, than whom we have few authorities better recognized in the history of Greek mathematics, maintains the common view that the lunes of Hippocrates are due to this writer, in spite of some doubt that has recently been cast upon it.

The summary which he gives of the work of the Greeks in geometry is, as would naturally be expected from him, very clear, and forms the principal portion of the work. The section on the early traces of the calculus is especially interesting, and refers of course to the contributions of Democritus, Eudoxus, and Archimedes, but without assigning the credit to Antiphon and Bryson that might be expected.

The mathematics of the Hindus is discussed with brief mention,—too brief, considering the contributions of the writers of India. The reader will not find the well-balanced judgment of the Oriental mathematics, nor the interesting information concerning its algebra, that he would expect from a perusal of the pages devoted to the work of the Greeks. The Arab contributions are somewhat more fully treated.

The medieval period in Europe is considered more at length, this being a period to which Professor Zeuthen has given much attention in his other works. Just at the present time, when Roger Bacon is much in the public eye, it is interesting to note that Professor Zeuthen dismisses him with exactly twelve words, and that these relate solely to optics.

It goes without saying that the book fills its purpose in a satisfactory manner. It gives a popular view of the progress of mathematics down to the period of the Renaissance, and is written in the pleasant style which characterizes all of the works of its distinguished author.

DAVID EUGENE SMITH.

Geometrie der Zahlen. Von HERMANN MINKOWSKI. Zweite Lieferung. 1910. viii + 15 pages. B. G. Teubner, Leipzig und Berlin.

THE first volume appeared in 1896. The preparation of the second volume was delayed by unexpected difficulties, and the author published in short articles in the journals most of the results initially intended for the second volume. These articles are accessible in the Gesammelte Abhandlungen of Minkowski, published in 1911 in two volumes, aggregating 872 pages. The present pamphlet is a continuation, bearing the same title, of the closing chapter of the first volume of Geometrie der Zahlen, and contains also a table of contents and index for the two volumes. This highly original contribution to the advanced part of the theory of numbers is already recognized to be of lasting importance. A more elementary introduction to the ideas there presented, as well as applications to algebraic numbers, are given in Minkowski's Diophantische Approximationen, which appeared in 1907 and was reviewed for the BULLETIN by the present writer, February, 1909 (volume 15, pages 251–252).

L. E. DICKSON.

Algebra of Quantics. By E. B. ELLIOTT. Second edition. Clarendon Press, Oxford, England, 1913. xvi + 416 pp.

THIS new edition of Professor Elliott's well-known work on algebraic theory presents more changes in typography than in content or method. Although the material has been considerably increased (by one eighth is the author's liberal estimate), the number of pages has been actually decreased. This has been accomplished by more compact printing, the adoption of a new method of writing fractions, and the insertion of many important equations in the body of the text instead of printing them in separate lines. These changes do not add to the appearance of the text, and surely not to the delight of the reader, to whom the outstanding equations were of great assistance in reference. The change from d to δ in writing partial derivatives strikes the eye at once. Paragraph numbers are practically unchanged.

Conservatism marks this new edition. Professor Elliott has not abandoned the English methods for German symbolism. In a majority of paragraphs there is no change, and most of the others have only slight verbal changes or an added sentence to clear up obscurities or to suggest further deductions. Chapters V and XV alone present important modifications. In Chapter V, on binary quantics, there is considerable rearrangement, and material has been added on "Invariants as functions of the differences of roots." A half dozen added pages are devoted largely to establishing the conditions under which $F(a_1, a_2, \cdots, a_p)$ can be expressed as a function of the differences of the arguments and to proving some properties of this function. The author notes that this should have been in the first edition. Chapter XV, on restricted substitutions, has undergone the most changes and contains the most new material. The half dozen paragraphs on Boolean systems for the linear form, the quadratic, the linear and quadratic, etc., have been replaced by one short chapter which suggests the method by which the Boolean systems for quantics of low order may be calculated. Ten pages have been added on the deserving topic of orthogonal concomitants. Both the absolute and non-absolute concomitants are treated for direct and skew orthogonal transformations. Complete systems of orthogonal invariants and covariants are given for the quadratic, cubic, and quartic. Much of this theory has been developed since the first edition, although some of the theorems date back to Sylvester.

Some other changes may be worth noting. In Chapter IV a few pages on German symbolism are added and the relation between it and hyperdeterminants is shown. In Chapter VII, dealing with the operators O and Ω, a long paragraph is inserted on "Separation of gradients into seminvariants and other parts," with examples. In Chapter VIII, on generating functions, an important article is added on a method of extracting generating functions for *invariants* only, and is illustrated by the sextic case. The real generating function for the sextic has been added to the table. In Chapter IX, we find a new paragraph on Hilbert's second proof of Gordan's theorem and a short one on syzygies and syzygants. Chapter XI contains a new proof of the theorem that every G is an ΩF, where G and F are both gradients, covering the case where F contains more advanced letters than G. The possible nature of F is brought out clearly in the discussion. Chapter XII contains a new article on apolarity and another on the canonical form $(1, 0, C, D, E, 0, 1)(X, Y)^6$ of the sextic. In Chapter XIV, the concomitants of the linear form and a p-ic are given and the most important of several additions in Chapter XVI is on "A generator of all covariant ternary sources."

The above with various minor additions contain scarcely enough of novelty to warrant the entry of this second edition upon a shelf already containing the first, although the added examples are of great value. In the preface the author expresses the hope that the work will continue to appeal to the beginner and further "trusts that the book will retain for some time to come a certain value to those . . .who will need to look back upon and utilize the labors of those pioneers of modern algebraic theory, who adorned the latter half of the nineteenth century, and have now passed away." The demand for a second edition makes it reasonable to expect that this hope will be realized.

D. D. LEIB.

Problèmes d'Analyse mathématique. By E. FABRY. Paris, Libraire Scientifique A. Hermann et Fils, 1913. 460 pp.

THE primary object of this collection of problems seems to be to furnish material in which French student candidates preparing for the examinations for certificates in differential and integral calculus might be interested. A list of problems (279) given at examinations within the last twelve years and at fourteen different examination centers scattered throughout France occupies the first 66 pages of the book. The detailed solutions of the problems proposed fill quite completely the remainder of the volume. The reviewer has checked a few—not many—of these solutions.

The 279 problems in the twelve chapters are distributed in numbers from 5 to 35 over the field of analysis as follows: Two chapters on quadratures and their geometric applications. One on line integrals (complex variable, analytic functions). Four on differential equations and their applications to plane curves, space curves, and surfaces, including the determination of geodesics, asymptotic lines, and lines of curvature. A chapter on ruled surfaces (developable). Two chapters on partial differential equations and their geometric applications. One each on total differentials and elliptic functions.

Some of the problems—a few—might be included in the lists of "harder problems" in calculus texts designed for the American college sophomore; more could be solved with much labor in algebraic and trigonometric reductions by a class in the advanced calculus, but many would undoubtedly have to be "left for later consideration." Certainly, if the candidate passed an examination on a similar list he might well feel that he deserved his certificate.

ERNEST W. PONZER.

Cours de Géométrie infinitésimale. By G. DEMARTRES. Paris, Libraire Gauthier-Villars, 1913. x + 455 pp.

EVEN a cursory inspection of this volume would lead one to conclude that it was never intended to rival the classic works of Bianchi or Darboux. One easily comes to the conclusion, especially after reading the excellent preface by Appell, that the book is intended to be a text, not a reference book to accompany lectures, on a subject which, according to the reviewer's opinion, is sadly neglected in mathematical courses offered in this country. A course on differential geometry after the plan of the book under review or that of Scheffer's "Anwendung der Differential und Integral-rechnung auf Geometrie," with which one is continually tempted to make comparisons, would help round out any undergraduate course in mathematics offered. Incidentally it would serve as a safe stepping-stone from the calculus to the maze of analysis into which the student too often is plunged headlong.

The correct pedagogical principle, that by far the majority of students feel constrained to make use of geometric interpretations of analytic processes in order to understand the latter thoroughly, is kept constantly in the foreground. In fact it is with this purpose in mind that the whole of the first part of the book (134 pages) is developed along entirely geometric lines. Here the geometry of the infinitesimal by no means plays the minor rôle it too often is assigned, if it is not neglected entirely—yes, even in treatises on geometry! The theory of infinitesimals and its application to the problems of curvature, order of contact, special curves, kinematics, etc., is developed geometrically. So also for space curves and surfaces. Definite notions are given concerning the nature of curvature, principal normal, tangent plane, osculating sphere, asymptotic and geodesic lines, etc., and theorems are proved entirely from the geometric side.

If it should be that the analysis because of its generality, rigor, and art of being conclusive appeals to the student with greater force, then he might well begin the book with the second part; for the last three of the four parts into which it is divided emphasize the analysis. Or he could compare the analytic treatment of the subject with the geometric given in Part I—a comparison which would certainly help fix its principles. However, as far as possible, an endeavor is made to hold fast to geometric interpretation no matter how analytical the treatment becomes. As an aid in this direction may be mentioned the general plan which is carried throughout the last three parts. This consists in expressing the coordinates of a moving point by a power series in terms of the displacement along a curve. A proper choice of the axes at the origin of the moving point allows a geometric interpretation of much of the analysis at each instant in terms of simple geometric relations.

The theory of lines, straight or curved, in a plane, free to move, or situated on a surface, together with the usual and some special theorems accompanying it, is developed in Parts II and III. In Part IV curvilinear coordinates are used in the problems and theorems of space curves and surfaces. Here is also included the application of line coordinates to the general theory of surfaces of the second order.

A list of 136 exercises, to some of which is added the suggestion that they be solved geometrically, is given at the close of the book.

It is to be regretted that the volume does not possess the high typographical tone and finish which characterize Scheffer's text.

ERNEST W. PONZER.

Untersuchungen über die im Schlusswort des Lie'schen Werkes "Geometrie der Berührungstransformationen" angedeuteten Probleme. Von PHILIPP ENGELHARDT (Inaugural-Dissertation, Würzburg). Leipzig, B. G. Teubner, 1910. 65 pp.

THE closing lines of Lie's Geometrie der Berührungstransformationen forecasted that a second volume, which never appeared, would treat the theory of differential equations from the standpoint of contact transformations and continuous groups. One may well hope that this field will soon receive its fair share of attention and be extensively developed beyond the elements established by Lie.

The thesis under review might be classed with fundamental literature having this tendency. The author's methods are, for the most part, those presented by Lie in chapter 14. The integral surfaces of a non-linear partial differential equation of the first order, $F(x, y, z, p, q) = 0$, bear an aggregate of ∞^3 exceptional curves called characteristics (Geometrie der Berührungstransformationen, pages 261, 498). Lie's summary of his last chapter is in the form of statements of six problems which are there solved. These are determinations of differential equations $F = 0$ defined by definite conditions imposed upon the characteristics or upon the normals to integral surfaces. Problem 2 is the determination of the equations $F = 0$ whose characteristics are lines of curvature upon all integral surfaces. Problem 4 is the determination of the equations $F = 0$ whose integral surfaces have for normals precisely lines of a given line complex. This thesis is devoted to twelve problems obtained by combining Lie's six problems. Thus problem (2, 4) of the thesis is the determination of all equations $F = 0$ whose characteristics form lines of curvature upon all integral surfaces and whose integral surfaces have for normals precisely lines of a given line complex. These prove to be equations having integral surfaces for which the complex of normals is the ∞^3 rays through a definite curve k. The characteristics are circles.

Geometrical intuition based upon configurations defined by Monge equations gives direction to the analysis. There is no introductory summary of the principles, although a desirable outline could have been given in a comparatively brief space and would have improved the exposition.

O. E. GLENN.

The Trisection of the Angle by Plane Geometry: Verified by Trigonometry with Concrete Examples. By JAMES WHITEFORD, B.A., M.D. Bowes and Bowes, 1911. 169 pp.

THE book under review is another illustration of how little the logic involved in the impossible construction proofs of Klein and others is appreciated by the ordinary mind. While actually quoting DeMorgan to the effect that the famous trisecting of the angle problem cannot be solved, yet the author claims to present a right line and circle solution of this famous problem. His confidence in his construction and proofs seems to rest upon the fact that they stand the test of trigonometry aided by logarithms in fifty concrete examples laboriously calculated to seven places of decimals. It is hardly necessary to say that he has not solved the famous trisection problem but simply exhibits a rather close approximation method. His construction is not even useful as a practical drawing-room approximation since simpler and more exact angle trisection methods are well known, although of course not euclidean methods. What a pity that so much time, patience, and labor should be wasted in such useless work!

ERNEST B. LYTLE.

Practical Algebra, Second Course. By J. V. COLLINS. American Book Company, 1912. x + 303 pp.

THIS book is intended to meet the needs of a text for a second course in algebra in such schools as offer a second course, often elective, in either the junior or senior year. After a rather rapid review of the usual topics of a first course in algebra, follow chapters on ratio and proportion, logarithms, arithmetical and geometrical progressions, the binomial theorem, inequalities, and exercises in physics.

The style is simple and the problems are numerous; the function notion is introduced and used in eight different places; the sine, cosine, tangent and cotangent functions are used in the solution of right triangles; correlation with geometry is made through problems involving geometric principles; graphic methods are used frequently; proper emphasis is placed upon translation from English to algebra and from algebra to English; historical notes and pictures of famous mathematicians appear in many places.

While the book as a whole seems to be a satisfactory text, yet we find a few points to criticise. Since ∞ is not a number symbol in elementary algebra, we dislike "The symbol ∞ denotes an exceedingly large number," etc., page 4. "The exponent 0 shows that x is used no times as a factor of the product, or has dropped out and so does not affect the product of the other factors," page 146, seems an unfortunate expression and not helpful in emphasizing the important fact that any number with a zero exponent is equal to unity. In the treatment of imaginaries it is feared that the student will miss the importance of expressing in the i-form before attempting operations with imaginaries. In variation the almost obsolete notation is used which many wish to see eliminated; the better equality form is mentioned but not sufficiently emphasized to avoid the difficulties students usually have with the language of variation. In the theory of the quadratic the general form $ax^2 + bx + c = 0$ seems much better and is more usual than the form $x^2 + px + q = 0$ which the author uses; $p^2 - 4q$ is certainly not the usual or common form of the discriminant of the quadratic.

ERNEST B. LYTLE.

Complete Business Arithmetic. By GEORGE H. VAN TUYL. New York, American Book Company, 1911. 4 + 416 pp.

ONE of the most serious problems which we have to face in the teaching of elementary mathematics at the present time relates to the work in our commercial courses. The world is far from coming to a decision as to what mathematics is best suited to the training of the boy and girl who propose to enter the field of commercial activity. The school of educators who believe in nothing but the immediately practical would admit only arithmetic, while those who believe that the person who is to succeed in any line of activity must have come in contact with all of the great fields of human interest, and in particular with mathematics, will demand at least some slight knowledge of algebra, geometry, and trigonometry.

In this country our schools of commerce have emphasized the work in arithmetic, rarely paying any attention to the adapting of algebra to their purposes, or giving any consideration to the important field of measurement. In this we are distinctly behind the schools of Germany, Switzerland, and Austria, to name the three countries which have given the most attention to the subject.

The work of Mr. Van Tuyl is typical of what we are doing in this country, and if it is not up to the ideal standard the blame is, to a great extent, due to the fact that we have not seriously attacked the problem of the curriculum. It seems very strange that a student who is using a book of this kind should need to be drilled upon the combinations in addition (page 31), or to have the simple process of multiplication explained to him (page 52), especially as the more difficult process of ordinary division is not explained at all. It is difficult to see why a student in commercial arithmetic should be spending his time in finding the prime factors of 12,464 (page 61), the greatest common divisor of 6,859, 11,191, and 3,610 (page 63), or the least common multiple of 36, 45, 72, 105, and 150 (page 64). In this era of the decimal fraction and of approximations, it is hard to justify the reduction of $\frac{3125}{}$ to an integer (page 72), or $1846\frac{3}{7}$ to an improper fraction (page 73). It would also trouble us to say what commercial interest is touched by such problems as the simplification of $\frac{7}{15}$ of $22\frac{1}{2} \div \frac{8}{13}$ of $\frac{26}{33}$. Why a subject like compound proportion (page 116), or even simple proportion as a method of solving commercial problems, and particularly with the old symbol (: :), should have place in a modern book like this it would be hard to tell, and it would be even more difficult to justify the expenditure of time necessary for dividing 8Kl. 7Hl. 5Dl. 7l. 6dl. 4cl. 5ml. by 5 (page 149).

On the other hand, the drill on the checking of computations, the work in graphs, and a considerable part of the work in elementary business principles have much to commend them.

One lays down the book with the feeling that our commercial courses in this country have not advanced as rapidly as our ordinary curriculum, and that, in spite of the good features mentioned above, this work will not help to improve the unfortunate situation.

DAVID EUGENE SMITH.

Text-book of Mechanics. By LOUIS A. MARTIN, JR. Vol. III: *Mechanics of Materials,* 1911, xiii + 229 pp. Vol. IV: *Applied Statics,* 1912, xii + 198 pp. 12mo. New York, John Wiley and Sons.

THE first two volumes of this text* deal with theoretical mechanics. The volumes under review, together with a projected volume on applied kinetics, will constitute a course in applied mechanics.

Volume III presents the usual theory of the deflection of beams by simple or compound stresses, statically indeterminate beams, struts and columns (eccentric loading, buckling, etc.), elastic failure, and envelopes. Also the principle that the external work done by the applied forces must equal the total resilience due to bending and shear is extensively used to determine deflections; and the equations of several elastic curves are very neatly found in this way. There is no treatment of curved bars, flat plates, or rotating disks, nor any mention of core sections in eccentrically loaded columns, but the problem of avoiding tension in such columns is treated briefly.

A fairly extensive range of applications is covered in the problems; but one is rather surprised at the omission of any applications to the distribution of steel in large reenforced concrete tanks, inasmuch as so many of these are now in use, at least in the west. Moreover, it might be well to include in a later edition a comparison of the various formulas for long columns as regards typical results, and also to insert a convenient collection of the most important tables and formulas. This would not hurt the book as a text and would make it more attractive for reference purposes.

The author says in the preface that he has attempted "to produce a book which will encourage the student to think and not to memorize, to do and not simply to accept something already done for him; but which still furnishes sufficient material in the way of explanation and example so that he will not become discouraged." He has succeeded remarkably well. There is no tendency toward "spoon-feeding"; yet the treatment is admirably clear and simple. Principles are concisely stated; and, with scarcely any exceptions, the exposition has been carefully thought out.

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* Reviewed in this BULLETIN, vol. 16 (1909), pp. 144–7.

It might be well to clarify the expression, "the inscribed curve to this polygon" (page 45), to explain why stresses may be regarded as constant over the surface of an infinitesimal element on page 137, while their increments had to be considered on page 135; to comment on the two apparently distinct meanings of the symbol $'M_x$ on pages 188–9; and to make the resilience formula (page 15) more general by changing the limits of integration to Δl_1 and Δl_2. Perhaps it would also be well, before asking the student to derive the equations for the elastic curve of a certain continuous beam (example 106), to suggest in some way the device of equating slopes of adjacent spans at an abutment to help determine the constants of integration.

Naturally mathematicians will object to speaking (pages 26, 32) of a discontinuity in a function or its graph at a point where the slope has a discontinuity but the graph itself is free from a break; also to the statement (concerning the vanishing of Q whenever M reaches a maximum) "as must evidently be the case from the relation $dM/dx = Q$." But let us refrain from throwing this last stone, until all of our standard texts on calculus have eliminated similar statements!

Of course, calculus is freely employed in this volume; and the method used in formulating differential relations is the usual one of considering an infinitesimal element and regarding some variable quantity as constant throughout the small element, and often incidentally dropping infinitesimals of higher orders. (To cite a simple case from volume IV, the differential of work, $dW = pdv$, is obtained by regarding p as a constant during the small change in v.) In teaching calculus to beginners this method is wholly unsatisfactory; but in making applications of the calculus it is far more convenient than any argument concerning limits, and is in almost universal use by practical men. Judging by our texts on calculus, is there not a gap right here in our teaching of the subject? Would it not be well toward the end of the calculus course to treat thoroughly this matter of formulating differential relations, with a detailed comparison of the two methods (which have been aptly called the rigorous and the vigorous), and with considerable practice on real applications? Possibly the present inadequate treatment of this matter is one cause of the difficulty which students generally have in connecting their calculus with their practical work.

But to return. Few errata have been noticed, and in most cases the correction is fairly obvious: read x^3 for x^2 (page 36, line 13); $\bar{y}$ for y (page 66, line 20); $\frac{\pi}{4}+\frac{\phi}{2}$ for $\frac{\pi}{2}+\frac{\phi}{2}$ in fig. 83 (page 160); *section* for *material* (page 177, line 23); *uniformly* for *gradually* (page 36, etc.); e for c in the answers to examples 113–116; and interchange t_1 and t_2 in the answer to example 119.

Volume IV reviews and supplements numerous sections in Volumes I, II; and carefully summarizes the theoretical mechanics presupposed for the applied work. The applications include dynamometers, expansion of steam, tractive power of locomotives, abutments, cranes, steam shovel and dipper dredge, cables, wedges, keys and cotters, sliding arms, screws, pivots, belts, journal friction, resistance to rolling, etc.

The treatment of solid statics presupposes no knowledge of the analytic geometry of space beyond the meaning of the three coordinates. (In fact it is surprising how little analytic geometry of the plane is used in many branches of applied mathematics, even where the calculus is freely employed. If we are interested in letting our students get early those tools which they will most use, it would seem desirable for us to postpone much of the analytic geometry, letting it follow an introductory course in calculus, and thus make room for early treatment of the latter subject.)

The author bases his entire treatment of the expansion of steam upon the assumption that Boyle's law applies; but he treats adiabatic expansion in the case of air. It might be well at least to remark that the indicator diagrams for steam engines often satisfy a law of the adiabatic type better than Boyle's law.

Methods of approximating plane areas are discussed; and Simpson's rules are given in their usual extended forms, rather troublesome to remember. Why not modify the statements, making the basis of each rule a formula for the area of a single strip,—e. g., $A = \frac{1}{8}(y_1 + 3y_2 + 3y_3 + y_4)l$ in the case of the three-eighths rule, where y_1 and y_4 are the bounding ordinates of the strip, and y_2 and y_3 are the ordinates trisecting the base (whose length is l). The general rule is, then, simply to divide the area into any number of strips and apply the above formula to each strip. Why bother to remember the result obtained by summing, and why base each rule upon a set of several strips and then have to remember whether the rule requires the total number of strips to be a multiple of two or three?

But this criticism, in common with several above, is directed more at current practice than at the particular volumes under review. In fact, the text is one of conspicuous merit, which not only promises to give good results, but also gives every indication of representing a course already taught with a high degree of success.

F. L. GRIFFIN.

Die angewandte Mathematik an den Deutschen mittleren Fachschulen der Maschinenindustrie. Von Dipl.-Ing. KARL OTT. Band IV, Heft 2, Internationale Mathematische Unterrichts-Kommission. Leipzig, Teubner, 1913. vi + 158 pp.

CONFESSIONS of faith are much in vogue these days with politicians, all of whom desire to be considered progressive. The author of the pamphlet under review uses considerable space, especially at the beginning, to create the proper atmosphere around the definite points of view from which the work of the particular schools on which he is reporting is to be judged. And in the confession of the principles of his faith he uses the words of Runge, of Göttingen, to express himself. Freely translated and condensed, these principles are about as given below.

The problems of applied science requiring mathematics call for quantitative results. Abstract and formal results are not sufficient. They must be correct to the proper decimal place. They should check. Accuracy and efficiency must obtain. Form and logical sequence must be seriously considered. . . . Suitable methods whether graphical, numerical, or mechanical are necessary. The province of applied mathematics is to furnish such suitable methods. And the field is not at all to be considered as apart from pure mathematics but a part of it. It is simply a question of a different state of mind handling its problems efficiently. To all of which the author adds in his own words that the engineering student must have a thorough understanding of the fundamental principles of the mathematics he applies and be able to reason clearly along technical lines.' However, he need not go into the full theory (abstract) of all the principles. An engineer may not be familiar with the writings of Carnot or Clausius, but he should be familiar with the thermodynamic principles they discussed and have had much experience in their application. Empirical methods and results arise and are well worth while. . . .

The schools under consideration arose because technical industry required them. They are strongly influenced by actual engineering practice, and of late years the Verein Deutscher Ingenieure has helped outline their programs and general engineering training.

The province of this special report is limited to those studies in a technical course in which the mathematics involved forms the back-bone. These are technical mechanics, including mechanics of materials, graphic statics, hydraulics, machine design, thermodynamics, and a general study of the problems of electrical engineering. The following is a composite program in a typical government school of the type under discussion.

First semester: Statics. Technical mechanics, with applications to mechanics of materials. Elements of graphic statics. 6 hours per week.

Second semester: Dynamics of rigid bodies. 6 hours per week.

Third semester: Hydraulics. Thermodynamics. 6 hours per week.

The methods of the seminar and laboratory play an important rôle in the instruction, while design as handled in this country meets with the author's warm approval.

Students enter differently prepared. The cry about the inability of some of their students to handle the fundamental principles of trigonometry, powers and roots, quadratics, logarithms, etc., sounds very much like the tale of woe uttered by our own instructors of freshman mathematics. Some schools require for entrance the equivalent of our customary first year mathematics, others offer a preparatory course. It takes a year, as with us, to obtain an efficient homogeneity of ability and preparation.

Some of the schools carry pure mathematics (analysis) and the applied mathematics at the same time; some try to correlate; others try the spiral method, where the principles are reviewed in the light of the principles of the calculus. With others the calculus is not studied as a separate subject and mechanics is handled by laboratory methods. In general the calculus is studied later than is the practice in our country and the whole plan of procedure is not outlined as definitely as with us. The case is worse in the private schools of similar kind. Undoubtedly the situation calls for instructors of ability to handle the applied mathematics—or to side-step its problems with agility.

The subject matter during the first half year must necessarily be quite elementary, and there are evidences of ingenious ways of dodging the rigors of the calculus. On the other hand there are equally ingenious demonstrations of a "near calculus" which in the nature of things must always arise in actual engineering problems. Definitions, approximations, experiments, mensurations, rules, formulas, instruments, models—all are used. Most excellent aids—all of them; but should they be first aids?

Examinations are given at the end of the course. These are partly oral, partly written. They are in charge of special committees of school officials and other dignitaries, including the instructors. The typical examination papers shown seem to be not at all as difficult as the average semester examinations on similar work in our technical schools.

A chapter on texts in use, their strong and weak points, is given. A list of reference works and journals is included. Slide rules, planimeters and models used in teaching are discussed. Typical methods of handling fundamental principles naturally calling for the calculus without its aid are illustrated. Graphical methods with special reference to graphic statics receive much encouragement.

Descriptive geometry here, as in our technical schools, aims to develop the students' power to see space relations. The curriculum seems not to include the mathematical treatment of the subject which so many critics have declared desirable.

ERNEST W. PONZER.

Astronomy, A Popular Handbook. By HAROLD JACOBY. The Macmillan Company. xi + 435 pages.

THIS book has a broader purpose than its title would seem to indicate. It is intended to be suitable not only for the general reader, but also for elementary college classes. As the author himself points out in the preface, these two purposes appear to be contradictory. While the general reader requires a treatment devoid of all mathematics, the college student should gain the additional insight into the subject made possible by the occasional use of algebra and geometry. The plan by which the author has met the difficulty is simple and natural. He has made the body of the text free of all mathematics, and has placed in an appendix such algebra and geometry as is used. Throughout the book it is apparent that he has aimed to make the presentation easy and attractive. The mathematical notes themselves are not at all difficult, and could certainly be mastered by any college student.

Although the author's clear and easy style of writing has enabled him to produce a book that is indeed interesting, it seems to me that upon certain questions he has given undue prominence to his own opinions, with a marked exclusion of many important facts. This is particularly unfortunate in a book intended largely for the general reader. The layman is apt to take the writer's words as those of an authority, and he, furthermore, has no way to know that he has read only a small side of a question. I refer to Professor Jacoby's treatment of the markings on Mars. In the discussion of this subject the book is in marked contrast to some of the excellent texts in general use, in which the strongest evidence in favor of, as well as against the canals, is plainly stated, and in which special emphasis is not placed on the writer's views. That Professor Jacoby's ideas concerning the canals are decided, and are emphatically expressed, is shown by the following statement, page 231: "We conclude that neither by visual nor by photographic evidence has the existence of an artificial network of markings been proven, or even rendered highly probable. Therefore the time has not yet come when we shall have to inquire whether geometric lines indicate the presence of intelligent inhabitants; that time will arrive if the lines themselves are ever shown to possess a real or even a highly probable existence." I do not think that I misunderstand the writer. It is true that the first sentence contains the word *artificial*; but the final statement leaves no doubt that it is the existence, not the origin of the lines, that the author is aiming at.

Professor Jacoby's treatment of the Martian question, and the character of the arguments that he advances, have already been considered at length by Mr. E. C. Slipher.* Mr. Slipher is an experienced observer, and has studied the planet under the most excellent conditions. I refer the reader to his paper for a fuller discussion than is given here. In view of the weight that might attach to a book by Professor Jacoby, it seems to me proper, however, to call attention in this place to some of the opinions that he advances.

It would seem that Professor Jacoby is an ultra-sceptic on the Martian question. As Mr. Slipher points out, he doubts even the universally accepted behavior of the polar caps. We read, page 225: "The polar spots *seem* to increase in the Martian winter season, and to diminish in the summer. *If so*, they may be ice caps, etc."† Small wonder then that he refuses to accept the existence of canals if there still lurks with him doubt concerning the behavior of the polar spots. If the agreement of observers since the time of Herschel is not sufficient to allow him to hold a more positive opinion regarding the caps, it is hard to imagine what form of proof he would require of the fainter markings.

As is the case with all those that deny the existence of canals, Professor Jacoby defends his position by advancing an illusion theory. In so doing he states certain views to which I do not think that observers, or scientists in general, will subscribe. On page 229 we find the following statements regarding the effect of training and experience in observing: "This theory explains why highly experienced observers see so much more than beginners. They think they are training the eye, so as to increase its powers, while in reality they may only be training that slight imperfection of the imagination which tends to increase details thought to be visible. . . .Nothing more strongly increases the powers of the imagination—of seeing the unseen—than the knowledge that others have already made the observation. We are very prone to 'see' what we are told by others is visible: we think we see what we desire and hope to see; do what we will, we cannot prevent this." It is not by advancing such theories that the Martian question will be settled. The decision lies with those who have observed the planet continuously under the best conditions that can be found; and the testimony of most of these is in strong agreement, both as to the character and extent of the markings. I imagine that Professor Jacoby regards all the details that he himself sees on Mars as real, and not the product of his imagination, even though some of them might not be seen by others, less experienced than himself.

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* "The Martian markings," in *Popular Astronomy*, March, 1914.

† The italics are mine.

It seems to me very unfortunate that statements, such as those quoted above, should be made in a popular book by a professor of astronomy. They are in no way calculated to increase the confidence of the public in the value of astronomic study. Perhaps the writer did not mean them to be as far-reaching as they seem. Doubtless he merely overstated himself in a moment of temporary enthusiasm on the Martian question; for we find a few pages later the following quotation from Herschel: "I determined to accept nothing on faith, but to see with my own eyes what others had seen before me." Or was this quotation given to discredit Herschel's work?—plainly he had the avowed purpose to "see" what he was "told by others was visible."

I shall mention an instance in which the author puts particular value in negative evidence. Professor Campbell's observations are taken as conclusive that water vapor does not exist on Mars. Why make no mention even of the fact that Mr. Very's measures of Dr. V. M. Slipher's spectrograms, without exception, show that water vapor is present? They were made with the best of instruments and under the most favorable conditions.

It is significant that the book contains no copy of any of the many beautiful drawings of Mars that show the canals. Was it not safe to gratify the natural curiosity of the reader, and allow him to see how so many have pictured those strange products of their imagination?

K. P. WILLIAMS.

NOTES.

THE annual meeting of the American Mathematical Society will be held in New York City on January 1–2, 1915. At this meeting President E. B. VAN VLECK will deliver his Presidential Address, on "The rôle of the point set theory in geometry and dynamics." The Chicago meeting of the Society will be held on December 28–29. Section A of the

American association for the advancement of science will hold its sessions on December 30–31, at Philadelphia.

A NEW edition of the List of Officers and Members of the American Mathematical Society is now in preparation, and will be issued in January. Blanks for furnishing information have been sent to members. A prompt response is requested, in order that the data may be correct and complete.

THE office of the American Mathematical Society, provided and partly furnished a year ago by Columbia University, was destroyed by fire on the early morning of October 10, with complete loss of files and records and a considerable stock of volumes of the BULLETIN and *Transactions*. Of the *Transactions* and the later volumes of the BULLETIN, the Society still has a sufficient supply. But the first ten volumes of the BULLETIN were completely destroyed. The Society will be glad to receive gifts of any of these early BULLETINS, and also copies of the Annual Register and Catalogue of the Library, which were also lost in the fire.

THE concluding (October) number of volume 15 of the *Transactions of the American Mathematical Society* contains the following papers: "The conic as a space element," by R. A. JOHNSON; "The Weierstrass *E*-function for problems of the calculus of variations in space," by G. A. BLISS; "The subgroups of the quaternary abelian linear group," by H. H. MITCHELL; "Transformations of conjugate systems with equal point invariants," by L. P. EISENHART; "Proof of the finiteness of the modular covariants of a system of binary forms and cogredient points," by F. B. WILEY; "On the degree of convergence of Sturm-Liouville series," by DUNHAM JACKSON; "Singular integral equations of the Volterra type," by C. E. LOVE; "On the reduction of integro-differential equations," by G. C. EVANS; "Invariants in the theory of numbers," by L. E. DICKSON. Also addenda and errata for volumes 11 and 14 by E. B. LYTLE and OSWALD VEBLEN, and general index of volumes 11–15.

THE concluding (October) number of volume 36 of the *American Journal of Mathematics* contains: "The quartic curve and its inscribed configurations," by H. BATEMAN; 'On the continuity of a Lebesgue integral with respect to a parameter," by J. K. LAMOND; "Geometry on ruled surfaces," by S. LEFSCHETZ; "Restricted systems of equations (second paper)," by A. B. COBLE; "Character of the solutions of certain functional equations," by T. E. MASON; "Binary conditions for double and triple points on a cubic," by L. A. HOWLAND; "Modular invariants of two pairs of cogredient variables," by W. C. KRATHWOHL.

THE opening (September) number of volume 16 of the *Annals of Mathematics* contains: "Invariantive characterization of some linear associative algebras," by OLIVE C. HAZLETT; "On the theory of n-lines," by LENNIE P. COPELAND; "Plane curves with consecutive double points," by F. R. SHARPE and C. F. CRAIG; "The effect of radiation on a small particle revolving about Jupiter," by T. H. BROWN; "Non-homogeneous linear equations in infinitely many unknowns," by ANNA J. PELL; "The inscribed and circumscribed squares of a quadrilateral and their significance in kinematic geometry," by C. M. HEBBERT; "A note on symmetric matrices," by G. A. BLISS.

AFTER ten years of successful experience, the Mathematical club of Syracuse University has been reorganized into a mathematical fraternity, Pi Mu Epsilon, whose aims are the advancement of mathematics and scholarship. The fraternity is incorporated under the laws of the State of New York, with power to grant charters to other chapters.

THE Association of mathematics teachers of New Jersey held its first meeting at Rutgers College on November 7. The programme included an address of welcome by President W. H. S. DEMAREST of Rutgers College and response by H. E. WEBB, and the following papers: "The place of mathematics in the high school curriculum," by G. H. ECKELS; "Number and the quadratic," by RICHARD MORRIS; "The mechanics of aviation," by L. P. EISENHART; "Advanced mathematics in the high school," by R. W. LORD; "High school mathematics," by T. G. VAN KIRK; "Fractional equations in algebra," by H. H. ALLEN; "Certain definitions in geometry," by H. E. WEBB. These were followed by the report of the committee on the constitution. The temporary officers of the society are RICHARD MORRIS, president; H. H. ALLEN, secretary-treasurer; G. G. BROWER and H. E. WEBB, executive committee.

UNIVERSITY OF MUNICH.—The following mathematical courses are announced for the present semester: By Professor F. LINDEMANN: Theory of functions of a complex variable, four hours; Plane analytic geometry, four hours; Theory of higher algebraic curves, two hours; Seminar, two hours.—By Professor A. VOSS: Differential calculus, four hours; with exercises, two hours; Mechanics, four hours; Seminar in mechanics, two hours.—By Professor A. PRINGSHEIM: Algebra, four hours; Theory of numbers, four hours.—By Professor H. BRUNN: Elements of higher mathematics, four hours.—By Professor F. HARTOGS: Descriptive geometry, I, four hours; with exercises, three hours; Theory of determinants, two hours.—By Dr. F. BÖHM: Integral calculus, with exercises, five hours; Mathematical statistics, four hours; Mathematical literature, two hours.—By Dr. H. DINGLER: Elementary mathematics, four hours; Differential geometry of plane curves, two hours; Colloquium, two hours; Reduction of observational data, two hours.—By Dr. A. ROSENTHAL: Theory of functions of a real variable, three hours; Seminar, two hours.

THE Prince Jablonowski Society announces the following prize problem for 1916:

To extend the theory of linear functional differential equations in any direction. It is particularly desired to have an exhaustive treatment of some new special cases.

ON the occasion of the Australian meeting of the British Association, the University of Adelaide conferred the degree of doctor of science on Sir OLIVER LODGE, Professor T. W. EDGEWORTH, of Oxford, and Professor E. W. BROWN, of Yale.

DR. W. VOGT, of the technical school at Carlsruhe, has been appointed professor of mathematics at the University of Heidelberg.

PROFESSOR R. ROTHE, of the technical school at Hanover, has been called to the technical school at Charlottenburg, to succeed the late Professor HETTNER.

At the University of Nebraska, Professor H. T. Johnson has resigned to enter business. Mr. O. Gish has been appointed instructor in mathematics.

Professor William Marshall, on leave from Purdue University, has returned from Europe and has been appointed assistant professor of mathematics in the University of Arizona for the year 1914–1915.

At the University of Pennsylvania, Dr. O. E. Glenn has been promoted to a full professorship of mathematics, and Dr. H. H. Mitchell to an assistant professorship of mathematics. Dr. L. J. Reed has been appointed instructor in mathematics.

At Wellesley College, Dr. Clara E. Smith has been promoted to an associate professorship of mathematics.

Dr. H. T. Burgess, of the University of Wisconsin, has been promoted to an assistant professorship of mathematics.

At the University of Nebraska, Dr. Henry Blumberg has been promoted to an assistant professorship of mathematics.

Professors P. Boutroux and J. H. M. Wedderburn, of Princeton University, have returned to Europe and offered their services to their respective governments.

Dr. C. Shaw has been appointed professor of mathematics in the University of Idaho.

Dr. C. F. Craig, of Cornell University, who was granted leave of absence during the present academic year, has resumed his regular duties at the University.

Mr. L. R. Ford, of Harvard University, has resigned, on account of the war, the Sheldon fellowship on which he was to have studied abroad, and has accepted a lectureship in mathematics at the University of Edinburgh.

Mr. P. E. Henke and Mr. L. E. Williams have been appointed instructors in mathematics at the Georgia School of Technology.

MR. W. A. REINERT has been appointed instructor in mathematics at the Michigan Agricultural College.

MR. H. D. FRARY has been appointed instructor in mathematics at the State University of Iowa.

MR. E. B. ESCOTT, of the University of Michigan, has resigned his position to enter on actuarial work.

BOOK catalogues: Galloway and Porter, Cambridge, England, catalogue 74, 412 titles.—Alexander Brunton, 54 Hanover Street, Edinburgh, catalogue of antiquarian and modern mathematical books.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ÅLANDER (M.). Sur le déplacement des zéros des fonctions entières par leur dérivation. Upsala, 1914. 8vo. 63 pp. M. 2.50

BACHELIER (L.). Le jeu, la chance et le hasard. (Bibliothèque de Philosophie scientifique.) Paris, Flammarion, 1914. 8vo. 320 pp. Fr. 3.50

BAYLISS (R. W.). A first school calculus. London, Arnold, 1914. Cr. 8vo. 4s. 6d.

BERNSTEIN (B. A.). Complete set of postulates for the logic of classes expressed in terms of the operation "exception", and a proof of the independence of a set of postulates due to Del Ré. Berkeley (*Univ. Calif. Publ.*), 1914. Royal 8vo. 10 pp.

BIOCHE (CH.). Histoire des mathématiques. Paris, E. Belin, 1914. 16mo. Fr. 1.75

BJÖRNBO (A.). See WERNER.

BOMPIANI (E.). Forma geometrica delle condizioni per la deformabilità delle ipersuperficie. Roma, tip. r. accademia dei Lincei (Estr. *Rendiconti d. r. accademia dei Lincei; scienze fisiche*), 1914. 8vo. 6 pp.

BRIDGES (J. H.). The life and work of Roger Bacon: an introduction to the Opus Majus. Edited with additional notes and tables by H. G. Jones. London, Williams and Norgate, 1914. 3s

CARMICHAEL (R. D.). The theory of numbers. (Mathematical Monographs, No. 13.) New York, Wiley, 1914. 8vo. 94 pp. Cloth. $1.00

CASTELNUOVO (G.). Lezioni di geometria analytica. 3a edizione. Milano, 1914. 8vo. L. 15.00

CHERUBINO (S.). Sulle curve iperellittiche con trasformazioni birazionali singolari in sè e sui loro moduli algebrici. Torino (*Atti della r. accademia delle scienze*), tip. V. Bona, 1914. 8vo. 22 pp.

COMBEROUSSE (C. DE). Cours de mathématiques. Tome IV: Algèbre supérieure. 2e partie: Etudes imaginaires. Théorie générale des équations. 4e édition. Paris, Gauthier-Villars, 1914. 8vo. 24+831 pp. Fr. 15.00

DE MORGAN (A.). Essays on the life and work of Newton. Edited, with notes and appendices, by P. E. B. Jourdain. Chicago, Open Court, 1914. 8vo. 212 pp. $1.25

EHRIG (G.). See FRIEDRICH (M.).

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome III, volume 4, fascicule 1: Quadriques, exposé d'après l'article allemand de O. Staude par A. Grévy. Paris, Gauthier-Villars, 1914. 8vo. 164 pp. Fr. 8.00

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band III 1, Heft 5, pp. 761–962: Elementare Geometrie vom Standpunkte der neueren Analysis aus, von J. Sommer. Elementargeometrie und elementare nicht-Euklidische Geometrie in synthetischer Darstellung, von M. Zacharias. 1ter Teil. Leipzig, Teubner, 1914. M. 6.00

FORSYTH (A. R.). A treatise on differential equations. 4th edition. London, Macmillan, 1914. 8vo. 584 pp. 14s.

FRIEDRICH (M.). Grundzüge der analytischen Geometrie. 3te Auflage von G. Ehrig. Leipzig, J. J. Weber, 1914. 207 pp. M. 2.50

GABRILOVITSCH (L.). Ueber mathematisches Denken und den Begriff der aktuellen Form. Berlin, 1914. M. 2.50

GARBE (E.). Zur Theorie der Integralgleichungen. (Diss.) Tübingen, 1914.

GIUDICE (F.). Teoria delle sezioni coniche e nozioni di geometria descrittiva. Pavia, 1914. 8vo. 218 pp. L. 3.50

GOURSAT (E.). Lehrbuch der Analysis. Nach der 2te französischen Auflage übersetzt von F. J. Schwartz. 3 Bande. Band I. Leipzig, 1914. 8vo. 12+591 pp. M. 12.00

GRÉVY (A.). See ENCYCLOPÉDIE.

GROMMER (J.). Ganze transzendente Funktionen mit lauter reelen Nullstellen. (Diss.) Göttingen, 1914.

JONES (H. G.). See BRIDGES (J. H.).

JOURDAIN (P. E. B.). See DE MORGAN (A.).

JUEL (C.). Einleitung in die Theorie der Elementarkurven dritter und vierter Ordnung. Kopenhagen, Høst, 1914.

KROHS (G.). Duhring und Fermat. Eine gemeinverständliche und mathematische Studie zum Wolfskehlpreise nebst einer Lösung des Fermat'schen Problems. Berlin, 1914. 4to. 27 pp. M. 1.00

LAUDIEN (H.). Entwicklung willkürlicher Funktionen nach Funktionen spezieller orthogonaler und biorthogonaler Systeme. (Diss.) Breslau, 1914.

LEHMER (D. N.). List of prime numbers from 1 to 10,006,721. (Carnegie Institution, Publication no. 165.) Washington, 1914. Oblong folio. 8+133 pp.

LÉVY (F.). Sur la détermination, par les fonctions elliptiques, du nombre des classes de formes quadratiques binaires de déterminant négatif donné. (Thèse, Zurich.) Genève, A. Kundig, 1914. 8vo. 48 pp.

Loewenberg (G.). Was muss man von der Differential- und Integralrechnung wissen? 2te Auflage. Berlin, 1914. 8vo.

Mansion (P.). Abriss der Theorie der Hyperbelfunktionen nebst einer rein analytischen Theorie der Kreisfunktionen. Nach der dritten französischen Auflage übersetzt. Leipzig, Teubner, 1913. 8vo. 44 pp. M. 1.25

Paterno (F. P.). Nuovi metodi d'analisi geometrografica. Part I. (*Atti del Collegio degli Ingegneri ed Architetti in Palermo.*) Palermo, Virzè, 1913. 8vo. 29 pp.

Pilizotti (K.). Geometrographische Untersuchungen einiger photogrammetrischer Probleme. Wien, 1912. 10 pp. M. 1.00

Royal Society. Catalogue of scientific papers, 1883–1900. Volume XIII, A-B. Cambridge, University Press, 1914. 98+951 pp. £2 10s.

Russell (B.). Our knowledge of the external world as a field for scientific method in philosophy. Chicago and London, Open Court, 1914. 8vo. 7+245 pp. 7s. 6d.

Salomon (C.). Questions inédites de magie arithmétique polygonale. Etoiles magiques à 8, 16 et 20 branches (24, 64 et 100 points) et rosaces hypermagiques (16, 25 et 36 points). Paris, Gauthier-Villars, 1913. 8vo. 22 pp. Fr. 1.50

Schwartz (F. J.). See Goursat (E.).

Schulz (O.). Ueber Tetraeder mit rationalen Masszahlen der Kantenlangen und des Volumens. Halle, 1914. 8vo. 12+292 pp. M. 4.50

Seidel (R.). Ueber starre raumliche Bewegungen, deren Achsenflächen Zylinder sind. (Diss.) Dresden, 1914.

Smolar (G.). Beweis des grossen Fermatschen Satzes über die Gleichung $a^n + b^n = c^n$. Brandeis a. E., 1914. 8vo. 25 pp. M. 2.00

Sommer (J.). See Encyklopädie.

Staude (O.). See Encyclopédie.

Terracini (A.). Alcune questioni sugli spazi tangenti e osculatori ad una varietà: nota I. Torino (*Atti della r. accademia delle scienze*), tip. V. Bona, 1914. 8vo. 36 pp.

Thompson (A. W. H.). A new analysis of plane geometry, finite and differential. Cambridge, University Press, 1914. 16+120 pp. 7s.

Tschuppik (W.). Ueber den Rotationskörper $X^2Z^2 + Y^2Z^2 = K$ als Integrator. Berlin, 1914. M. 0.70

Uccelli (A.). I fondamenti della geometria di posizione. (Biblioteca del popolo no. 566.) Milano, casa ed. Sonzogno (Matarelli), 1914. 16mo. 56 pp. L. 0.20

Voss (A.). Ueber die mathematische Erkenntnis. (Die Kultur der Gegenwart. Des Gesamtwerkes Teil III Abteilung I: Die mathematischen Wissenschaften, unter Leitung von F. Klein. 3te Lieferung.) Leipzig, Teubner, 1914. M. 5.00

Werner or Verner. Ioannis Verneri de triangulis sphaericis libri quatuor. De meteoroscopiis libri sex. Cum prooemio Georgii Ioachimi Rhetici. II: De meteoroscopiis, herausgegeben von Joseph Würschmidt unter Benutzung der Vorarbeiten von Dr. A. Bjørnbo. Leipzig, Teubner, 1913. 8vo. 2+260 pp. M. 12.00

Wilensky (M.). Ueber Besselsche Funktionen. (Diss., Bern.) Zurich, Leemann, 1914. 8vo. 65 pp.

Würschmidt (J.). See Werner.

Zacharias (M.). See Encyklopädie.

II. ELEMENTARY MATHEMATICS.

Amaturo (M.). Alcuni teoremi sui triangoli. Napoli, tip. r. accademia delle scienze fisiche e matematiche (Estr. *Archivo del ministero della P. I.*, 1885), 1913. 8vo. 19 pp. con tavola.

Barriol (A.). See Trignart (A.).

Braginton (W.). Arithmetic papers for junior pupils. With or without answers. London, Macdonald and Evans, 1914. 9d.

Caronnet (T.). Cours de trigonométrie à l'usage des candidats au Baccalauréat, à l'Ecole spéciale militaire de Saint-Cyr et à l'Institut agronomique. Paris, Gauthier-Villars, 1912. 8vo. 218 pp. Fr. 4.50

Chignell (N. J.) and Paterson (W. E.). Arithmetic. Oxford, Clarendon Press, 1914. Part I, 2s. 6d. Part II, 2s. 6d.

Davison (C.). Mathematical problem papers for secondary schools. London, Bell, 1914. 8vo. 2s. 6d.

Deckert (A.). Lehrbuch der Mathematik. Breslau, Goerlich, 1913–1914. Algebra und Planimetrie für Präparandenanstalten. 208 pp. Geb. M. 3.30. Algebra fur Seminare. 214 pp. Geb. M. 2.50. Planimetrie, Stereometrie und Trigonometrie für Seminare. 214 pp. Geb. M. 3.50. (Daraus als Sonderausgabe: Stereometrie. 91 pp. Geb. M. 1.75. Trigonometrie. 80 pp. Geb. M. 1.75.)

Druxes (J.). Ausführlicher Lehrgang der Arithmetik und Algebra nach modernen Grundsatzen, mit den vollstandigen Lösungen der Aufgaben der "Sammlung von Beispielen und Aufgaben von Heis." Für Lehrer und Studierende. Bearbeitet unter Mitwirkung von G. Hoffmann. 2ter Band: Gleichungen, graphische Darstellungen, Nomographie. Koln, Du Mont-Schaubers, 1913. Geb. M. 10.00

Emmerich (A.). Kubische Aufgaben aus der Stereometrie. Nebst Anleitungen zur Losung, Determinationen und den Ergebnissen der Zahlenbeispiele. Weinheim, Ackermann, 1914. 127 pp. Geb. M. 2.60

Fischer (B.). Anschauungsmittel im mathematischen Unterricht. Leipzig, Teubner, 1913. Gr. 8vo. 40 pp. M. 0.60

Galdéano (Z. G. de). Annuario de propaganda matemática (1914) comprendiendo el curso de extension universitaria Genesis y desenvolvimento matematico. Cuaderno primero. Saragossa, G. Casañel, 1914. 8vo. 63 pp.

Henkler (P.) und Körner (K.). Arithmetik und Geometrie für die oberen Klassen der Lehrerbildungsanstalten. Berlin, Union Deutsche Verlagsgesellschaft, 1914.

Hoffmann (G.). See Druxes (J.).

Hupe (A.). See Müller (H.).

Körner (K.). See Henkler (P.).

Kullrich (E.). See Müller (H.).

Lauri (G.). Nuovo trattato di trigonometria piana e studio analitico di tutti i valori trigonometrici finora conosciuti. Torino, 1914. 8vo. 244 pp.+188 tavole.

——. Serie di 8 nuove tavole trigonometriche. Torino, 1914. 8vo.

——. Specchio numerico dei valori che figurano nelle Nuove tavole trigonometriche. Torino, 1914. 8vo. 24+102 pp.

Lietzmann (W.). See Schriften.

Müller (H.), Hupe (A.) und Kullrich (E.). Die Mathematik auf den Gymnasien und Realschulen. Ausgabe B: Für reale Anstalten und Reformschulen. 2ter Teil: Oberstufe. Abteil. II: Geometrie der Kegelschnitte, Darstellende Geometrie. 4te Auflage. Leipzig, Teubner, 1914. 216 pp. Geb. M. 2.40

Paterson (W. E.) and Taylor (E. O.). Elementary geometry, theoretical and practical. Volume I: Triangles and quadrilaterals. Oxford, Clarendon Press, 1914. 1s. 8d.

——. See Chignell (N. J.).

Scharrelmann (H.). Produktive Geometrie. Hamburg, Janssen, 1914. 108 pp. Geb. M. 2.80

Schriften des deutschen Ausschusses für den mathematischen und naturwissenschaftlichen Unterricht. Heft 18: Bericht uber die Tatigkeit des deutschen Ausschusses im Jahre 1913, erstattet vom W. Lietzmann. Leipzig, Teubner, 1914. 8vo. 10 pp.

Taylor (E. O.). See Paterson (W. E.).

Trignart (A.). Table auxiliaire d'interêts composés avec une préface de A. Barriol. Paris, Gauthier-Villars, 1914. 8vo. 8+24 pp. Fr. 2.00

Unger (F.). Der Rechenunterricht auf allen Stufen unter Ausschluss der ersten drei Schuljahre. Ein Handbuch für Lehrende. Leipzig und Berlin, J. Klinkhardt, 1914. 6+418 pp. M. 7.20

Warren (A. T.). Experimental and theoretical course of geometry. 4th edition, revised. Oxford, Clarendon Press, 1911. 12mo. 8+298 pp. Cloth. 2s.

III. APPLIED MATHEMATICS.

Almansi (E.). Le equazioni generali della dinamica e la legge di gravitazione. Roma, 1913. 4to.

Annuaire de l'Observatoire royal de Belgique, publié sous la direction de G. Lecointe, 1914. Brussels, 1913. 8vo. 8+506 pp.+5 pls. Cloth. Fr. 3.00

Autenrieth (E.). Technische Mechanik. Ein Lehrbuch der Statik und Dynamik für Maschinen- und Bauingenieure. 2te Auflage. Neu bearbeitet von M. Ensslin. Berlin, 1914. M. 18.00

Avigdor (A.). Riserve matematiche. Calcolo delle riserve matematiche per tassi non contenuti nelle tavole e loro variazioni al variare del tasso. (Dissertazione di laurea, R. istituto superiore di studi commerciali in Torino.) Torino, Viretto, 1913. 8vo. 34 pp.

Ball (L. De). Lehrbuch der sphärischen Astronomie. Leipzig, Engelmann, 1912. 16+388 pp. Sm. 4to. Geb. M. 21.50

Bartoli (G.). Scienze cosmica. Il mistero dell' universo e la scienza. Torino, 1914. 12mo. 342 pp. M. 3.50

Bauditsch (J.). See Grossl (R.).

Billows (E.). Modificazioni al goniometro Babinet per misurare angoli rientranti e per misurare cristalli molto piccoli. Padova, Società coop. tipografia, 1914. 8vo. 7 pp.

Block (W.). See Thomson (W.).

Briggs (W.) and Bryan (G. H.). Matriculation mechanics. 9th impression (3d edition). London, University Tutorial Press, 1914. 8vo. 8+363 pp. 3s. 6d.

Bryan (G. H.). See Briggs (W.).

Caspar (—.). Technologisches Wörterbuch, Deutsch-Englisch und Englisch-Deutsch. Umfassend die wichtigsten Wörter und Ausdrücke, angewandt in Technologie, Ingenieurwesen, Maschinenbau u. s. w. Milwaukee, Wise, 1914. 8vo. 264 pp. Cloth. $1.00

Cotton (E.). Cours de mécanique générale. (Bibliothèque de l'élève ingénieure.) Grenoble, Rey, and Paris, Gauthier-Villars, 1914. 8vo. 166 pp. Fr. 5.00

Dales (J. H.). A manual of mechanical drawing. Cambridge, University Press, 1914. 8vo. 3s.

Fawdry (R. C.). Statics. Part I. London, Bell, 1914. 7+165 pp. 2s. 6d.

Flammarion (C.). Astronomy. London, Constable, 1914. 8vo. 9+191 pp. 2s.

Fraenckel (A.). Theorie der Wechselströme. Berlin, 1914. M. 10.00

Freeman (N. H.). Baumé and specific gravity tables for liquids lighter than water. London, Spon, 1914. 2s.

Galitzin (B.). Vorlesungen über Seismometrie. Deutsche Bearbeitung, unter Mitwirkung von Clara Reinfeldt herausgegeben von O. Hecker. Leipzig, 1914. M. 22.00

Gilbert (L.). Das Relativitatsprinzip, die jungste Modenarrheit der Wissenschaft. Und die Losung des Fizeau-Problems. Wissenschaftliche Satyren. Band I. Brackwede i. W., 1914.

Ginzel (F. K.). Handbuch der mathematischen und technischen Chronologie. Das Zeitrechnungswesen der Völker. Band III: Zeitrechnung der Makedonier, Kleinasier, u. Syrer, der Germanen u. Kelten des Mittelalters, der Byzantiner (u. Russen), Armenier, Kopten, Abyssinier; Zeitrechnung der neueren Zeit. Nachtrage zu Bd. I–III. Leipzig, 1914. Gr. 8vo. 7+445 u. (Tabellen) 44 pp. M. 16.00

Grossl (R.) und Bauditsch (J.). Mechanik der flüssigen Körper. Lehrbuch für hohere Gewerbeschulen und verwandte Lehranstalten. Wien, 1914. Gr. 8vo. 7+240 pp. M. 6.00

Günther (S.). Kosmo- und geophysikalische Anschauungen eines vergessenen Baierischen Gelehrten. München, 1914. Gr. 8vo. 34 pp. M. 1.00

Guyau (A.). Le téléphone, instrument de mesure. 2e édition. Paris, Gauthier-Villars, 1914. 16mo. 6+150 pp. Fr. 2.75

Hagen (J. G.). Die veränderlichen Sterne. 2 Bände. Bd. I: Geschichtlich-technischer Teil (4 Lieferungen). Liefg. 2: Beobachtung der veranderlichen Sterne. Freiburg, 1914. Gr. 4to. Pp. 4+153–334. M. 10.00

HECKER (O.). See GALITZIN (B.).

HENSCHKE (E.). Ueber eine Form des Prinzips der kleinsten Wirkung in der Elektrodynamik des Relativitätsprinzips. (Diss.) Leipzig, 1912. 8vo. 88 pp. M. 2.00

IKLÉ (M.). See ZEEMAN (P.).

JACKSON (C. S.) and ROBERTS (W. M.). A book of elementary mechanics. London, Dent, 1914. 8vo. 3s. 6d.

JACOBY (H.). Astronomy. A popular handbook. London, Macmillan, 1913. 8vo. 450 pp. Cloth. 10s. 6d.

KIPPAX (J. R.). The call of the stars. New York, Putnam, 1914. 18+431 pp. $2.50

KRÜGER (L.). Konforme Abbildung des Erdellipsoids in der Ebene. (Veröffentlichung des k. Preussischen Geodätischen Institutes. Neue Folge, Nr. 52.) Leipzig, Teubner, 1912. 4to. 10+172 pp. M. 9.00

LECOINTE (G.). See ANNUAIRE.

LUDWIG (W.). Lehrbuch der politischen Arithmetik. 3te Auflage. Wien, 1914. Gr. 8vo. 4+233 pp. M. 3.40

MICHAILOW (A. A.). Outlines of theoretical astronomy (in Russian). St. Petersburg, 1914. 8vo. $1.50

NÄBAUER (M.). Grundzüge der Geodäsie und Astronomie. (Handbuch der angewandten Mathematik hrsg. von H. E. Timerding.) Leipzig, Teubner, 1914.

NERNST (W.). The theory of the solid state. London, Hodder and Stoughton, 1914. 8vo. 8+104 pp. 2s. 6d.

PALÁGYI (M.). Die Relativitätstheorie in der modernen Physik. Berlin, 1914. M. 1.50

PANNELT (J. R.). See STANTON (T. E.).

PEARSON (K.). Tables for statisticians and biometricians. Cambridge, University Press, 1914. 4to. 9s.

PETROVITSCH (S.). Course of rational mechanics. IV: Dynamics of a rigid body (in Russian). St. Petersburg, 1913. 8vo. 4+339 pp.

PRICE (E. W.). The essence of astronomy. New York, Putnam, 1914. 8vo. 14+207 pp. Cloth. $1.00

REINFELDT (C.). See GALITZIN (B.).

ROBERTS (W. M.). See JACKSON (C. S.).

SCHACHENMEIER (R.). Zur mathematischen Theorie der Beugung an Schirmen von beliebiger Form. Karlsruhe, G. Braun, 1914.

STANLEY (H.). Practical science for engineering students. London, Methuen, n. d. 7+166 pp. 3s.

STANTON (T. E.) and PANNELT (J. R.). Similarity of motion in relation to the surface friction of fluids. London (*Philosophical Transactions*), 1914. 4to. 26 pp. 2s. 6d.

THOMÄLEN (A.). Kurzes Lehrbuch der Elektrotechnik. 6te Auflage. Berlin, 1914. 8vo. Leinenband. M. 12.00

THOMSON (SIR J. J.). The atomic theory. (Romanes Lecture, 1914.) Oxford, University Press, 1914. 8vo. 39 pp. 1s. 6d.

THOMSON (W.). Ueber die dynamische Theorie der Wärme mit numerischen Ergebnissen aus Herrn Joules Aequivalent einer thermischen Einheit und Herrn Regnaults Messungen am Dampf. Ins Deutsche ubertragen und herausgegeben von W. Block. Leipzig, 1914. M. 5.20

TIMERDING (H. E.). See NÁBAUER (M.).

TOMPKINS (A. E.). Marine engineering. 4th edition, revised and partly rewritten. Illustr. 8vo. London, Macmillan, 1914. 15s.

VÉRONNET (A.). La forme de la terre et sa constitution interne. Paris, Hermann, 1914. 32 pp.

——. Les hypothèses cosmogoniques modernes. Paris, Hermann, 1914. 3+171 pp.

ZEEMAN (P.). Magnetooptische Untersuchungen mit besonderer Berücksichtigung der magnetischen Zerlegung der Spektrallinien. Deutsch von Max Iklé. Leipzig, J. A. Barth, 1914. 8vo. 12+242 pp. M. 8.00

L RARY

THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and seventy-second regular meeting of the Society was held in New York City on Saturday, October 31, 1914, extending through the usual morning and afternoon sessions. The attendance included the following thirty-eight members:

Mr. A. A. Bennett, Professor W. J. Berry, Dr. Emily Coddington, Professor F. N. Cole, Dr. Louise D. Cummings, Dr. H. B. Curtis, Professor L. P. Eisenhart, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Dr. G. M. Green, Professor C. C. Grove, Professor H. E. Hawkes, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. P. H. Linehan, Professor James Maclay, Professor Helen A. Merrill, Dr. R. L. Moore, Professor Richard Morris, Mr. G. W. Mullins, Professor W. F. Osgood, Dr. G. A. Pfeiffer, Dr. H. W. Reddick, Miss C. E. Seely, Professor L. P. Siceloff, Professor Clara E. Smith, Professor P. F. Smith, Professor W. B. Stone, Professor H. D. Thompson, Professor Oswald Veblen, Mr. H. E. Webb, Mr. R. A. Wetzel, Professor H. S. White, Miss E. C. Williams, Professor E. B. Wilson.

Vice-President L. P. Eisenhart occupied the chair. The Council announced the election of the following persons to membership in the Society: Dr. H. R. Kingston, University of Manitoba; Dr. Edward Kircher, Massachusetts Institute of Technology; Mr. Colin MacLennan, Havana Railway, Light and Power Company; Mr. E. E. Moots, Walla Walla, Wash.; Mr. C. N. Reynolds, Jr., Harvard University; Dr. Joseph Rosenbaum, New Haven, Conn.; Dr. Joseph Slepian, Cornell University; Dr. Anna H. Tappan, Iowa State College; Dr. Mabel M. Young, Wellesley College. Four applications for membership in the Society were received.

The Council submitted a list of nominations for officers and other members of the Council, to be placed on the official ballot for the annual election. A committee was appointed to audit the accounts of the Treasurer for the current year. Arrangements were made to adjust the insurance on the property of the Society destroyed by fire on October 10, and to restore the office to working conditions.

The following papers were read at this meeting:

(1) Dr. G. M. Green: "On completely integrable systems of homogeneous linear partial differential equations."

(2) Dr. G. A. Pfeiffer: "Contributions to the conformal geometry of analytic arcs."

(3) Dr. C. A. Fischer: "Conditions for a minimum of an n-fold integral."

(4) Mr. E. C. Kemble: "Note on the definition of work."

(5) Professor H. S. White: "Census of the triad systems on 15 letters."

(6) Professor Edward Kasner: "A law of reciprocity in the calculus of variations."

(7) Professor K. P. Williams: "Concerning a certain totally discontinuous function."

(8) Professor T. H. Gronwall: "Some remarks on conformal representation."

The paper of Mr. Kemble was communicated to the Society and read by Professor Osgood. In the absence of the authors, the papers of Professors Williams and Gronwall were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The completely integrable systems of homogeneous linear partial differential equations considered in Dr. Green's paper contain any number m of dependent variables $y_1, y_2, \cdots, y_m$, and any number p of independent variables $u_1, u_2, \cdots, u_p$. Any solution y_k of the system is expressible linearly, with constant coefficients, in terms of a fundamental system of n solutions: $y_k = c_1 y_k^{(1)} + c_2 y_k^{(2)} + \cdots + c_n y_k^{(n)}$; and all derivatives of the dependent variables, of any order, are expressible linearly in terms of n primary derivatives. A certain determinant W, formed from the primary derivatives of a fundamental system of solutions, takes the place in this discussion of the wronskian for a single ordinary homogeneous linear differential equation of the nth order. It is proved that the first derivatives of W with respect to the independent variables are given by the equations

$$W_{u_1} = f_1 W, \quad W_{u_2} = f_2 W, \quad \cdots, \quad W_{u_p} = f_p W,$$

where the f's are functions of the coefficients, and possibly of some of their derivatives, of the given system of differential equations. They satisfy the integrability conditions

$\partial f_s/\partial u_t = \partial f_t/\partial u_s$, so that W may be found by a quadrature from the coefficients of the differential equations.

The chief purpose of the paper is to prove that the system of differential equations may always be reduced, and without any difficulty, to a canonical form, in which the f's are all zero. The coefficients of the canonical form are expressions in the coefficients of the original system of differential equations, and are such that they remain unchanged for any transformation of the dependent variables of the form $y_1 = \lambda_1\bar{y}_1, \cdots, y_m = \lambda_m\bar{y}_m$, where the λ's are arbitrary functions of the independent variables $u_1, \cdots, u_p$. It is important, in Professor Wilczynski's method for investigations in projective differential geometry, to find the above-mentioned seminvariant functions, and the present paper gives a method, always applicable, for calculating them expeditiously.

2. The greater part of Dr. Pfeiffer's paper is devoted to an investigation of the properties of symmetry, i. e., the pairing of points symmetric with respect to an analytic arc, and the associated functional equations. Theorems are given concerning the existence of an analytic arc such that with respect to it one of two given intersecting analytic arcs is the symmetric image of the other. This is "the bisection problem," so called by Kasner. A theorem concerning the "n-sector of two given intersecting analytic arcs" is also obtained.

Closely related results are also given stating necessary and sufficient conditions for the conformal equivalence of certain pairs of intersecting analytic arcs to a rectilinear angle and showing the existence of a unique absolute conformal invariant of a pair of intersecting analytic arcs such that the magnitude of the angle formed by them is commensurable with π. Kasner has already shown that at least one such invariant exists and proved its uniqueness in a few special cases.

3. The necessary conditions for a minimum of a double integral expressed in parametric form have been discussed by Kobb and others. In this paper Dr. Fischer derives the analogous conditions for a minimum of the n-fold integral

$$\int\int \cdots \int F(x_1, x_2, \cdots, x_{n+1}, p_{11}, p_{12}, \cdots, p_{n+1n})du_1\, du_2 \cdots du_n.$$

The analogue of the Lagrange differential equation, and a variable boundary condition are derived. The variable boundary condition is reduced to a very simple form in the special case where $x_1, x_2, \cdots, x_n$ are the independent variables. Then the analogues of the Legendre and Jacobi conditions are discussed, and sufficient conditions are derived for a permanent sign for the second variation.

4. In Mr. Kemble's paper a difficulty in the usual definition of the work done by a force applied at a point is indicated, and a new definition is suggested. It is proved that the work done as defined is equal to the increase in the kinetic energy of the body to which the force is applied.

5. Special methods from 1852 to 1913 brought to light 12 distinct triad systems on 15 letters. Miss L. D. Cummings added, in 1914, 12 further systems. No attempt had been made to secure an exhaustive list of possible systems. In the present paper Professor White studies first the kind of substitution that can leave a triad system invariant. All such must have as some power a substitution of one or another of seven typical forms. The problem therefore is next to find all systems that are invariant under each of these seven substitutions. In this way only such systems are omitted as may have no group except identity. By this means twenty new systems are found.

6. In a previous paper, Professor Kasner investigated the geometric character of the transversality relations connected with the minimizing of double and simple integrals in space. The main result of the present paper is that with each double integral $\int\int F(x, y, z, p, q)dxdy$ there is associated a simple integral $\int G(x, y, z, y', z')dx$ such that the corresponding transversalities are inverse to each other. Each of the integrand functions F and G determines the other except for a factor involving only x, y, z. Other theorems relate to the case where F and G have the same form, and the case where the two transversalities are orthogonally related.

7. Professor Williams's paper appeared in full in the December BULLETIN.

8. In the present paper, Professor Gronwall proves the following theorems:

I. When the analytic function $z = f(x) = a_0 + a_1x + \cdots + a_nx^n + \cdots$ effects the conformal representation of the circle $|x| < 1$ on a simple (that is, simply connected and nowhere overlapping) region in the z-plane, the area of this region not exceeding A, then, for $|x| \leqq r < 1$,

$$|f(x)| \leqq \sqrt{\frac{A}{\pi} \cdot \log\frac{1}{1 - r^2}} \quad \text{and} \quad |f'(x)| \leqq \sqrt{\frac{A}{\pi}} \cdot \frac{1}{1 - r^2},$$

and these upper boundaries of $|f(x)|$ and $|f'(x)|$ cannot be replaced by any smaller ones. Less accurate limitations have been given by Koebe and Courant.

II. When $z = f(x) = 1/x + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$ effects the conformal representation of the circle $|x| < 1$ on a simple region in the z-plane containing the point at infinity, then $|f(x)| < 9/4r$ for $|x| = r < 1$. A less accurate limitation has been given by Fricke.

F. N. COLE,
Secretary.

THE TWENTY-SIXTH REGULAR MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-sixth regular meeting of the San Francisco Section of the Society was held at the University of California on October 24, 1914. Twenty-two persons were present, including the following members of the Society:

Professor R. E. Allardice, Dr. B. A. Bernstein, Professor H. F. Blichfeldt, Professor C. E. Brooks, Dr. Thomas Buck, Professor L. E. Dickson, Professor M. W. Haskell, Professor L. M. Hoskins, Dr. Frank Irwin, Professor D. N. Lehmer, Professor J. H. McDonald, Professor W. A. Manning, Professor H. C. Moreno, Professor C. A. Noble, Professor E. W. Ponzer.

The chairman of the Section, Professor Manning, presided at the opening of the meeting; the chairman-elect, Professor Haskell, then took the chair. The following officers were elected for the ensuing year: chairman, Professor Haskell; secretary, Dr. Buck; programme committee, Professors Manning and Blichfeldt, and Dr. Buck.

It was voted to omit the spring meeting of the Section because of the summer meeting of the Society to be held in San Francisco, and to hold the fall meeting at Stanford University at a date to be determined later.

The question of a colloquium in connection with the summer meeting in 1915 was discussed, and it was decided to petition the Council to arrange for such a colloquium.

The members took luncheon together as usual.

The following papers were presented at this meeting:

(1) Dr. FRANK IRWIN: "Relation between the zeros of a rational integral function and its derivative."

(2) Professor M. W. HASKELL: "The maximum number of stationary points of algebraic curves."

(3) Professor D. N. LEHMER: "Some further results in the theory of the continued fraction representing the surd $R^{1/\lambda}$" (second preliminary report).

(4) Professor L. E. DICKSON: "On formal and modular invariants."

(5) Professor L. E. DICKSON: "On modular curves."

Abstracts of the papers follow below.

1. The proposition that the roots of the derivative of a polynomial lie, in the complex plane, inside the smallest convex polygon containing the roots of the polynomial itself is well known; but Hayashi's proof recently published in the *Annals of Mathematics* is the first, so far as known, not based on dynamical conceptions. Dr. Irwin points out that an extremely simple proof of this kind may be given.

2. Professor Haskell shows that the maximum number of cusps possible for a plane curve of order m is the greatest integer in $m(m-2)/3$; and that in every case there is a self-dual curve with this maximum number of cusps. The cases $m = 4$ and $m = 6$ are exceptions, the maximum number of cusps being 3 and 9 respectively.

In the case of space curves, the Plücker equations can be satisfied by the same maximum number of stationary points, but it can be shown that the curves are then either plane curves or reducible.

3. Representing the complete quotient in the expansion of $R^{1/\lambda}$ in a continued fraction by

$$\frac{P_n + A_n R^{1/\lambda} + B_n R^{2/\lambda} + C_n R^{3/\lambda} + \cdots + K_n R^{(\lambda-2)/\lambda} + L_n R^{(\lambda-1)/\lambda}}{Q_n}$$

and the corresponding convergent by α_n/β_n, Professor Lehmer has obtained certain interesting inequalities connecting the P's and Q's, which show that there can be only a finite number of P's which have the same value. In a former paper it was shown that the Q's satisfy the indeterminate equation

$$(-1)^{n-1} Q_n = \alpha_n^\lambda - R\gamma_n^\lambda,$$

and by a general theorem due to Axel Thue (Christiania, *Videnskabs-Selskabet Skrifter*, 1908, No. 3), there can be only a finite number of Q's having the same value in the expansion. This important result has not yet been derived from the discussion of the continued fraction itself.

4. The first paper by Professor Dickson gave a survey of the main results in the theory of invariants arising in the theory of numbers. Special attention was given to the construction of formal modular invariants from the geometrical standpoint developed in the October number of the *Transactions*.

5. The second paper by Professor Dickson related to the theory of modular cubic and quartic curves for the interesting case in which the modulus is 2. Such a quartic curve has at most seven bitangents (and aside from special cases exactly seven) whose intersections are either singular points or points with indeterminate polars. In general, all such points are intersections of bitangents. The equivalence of two quartic curves can be decided from a knowledge of their real points, their singular points, and their points with indeterminate polars.

THOMAS BUCK,
Secretary of the Section.

MODULAR INVARIANT PROCESSES.

BY PROFESSOR O. E. GLENN.

(Read before the American Mathematical Society, September 8, 1914.)

Introduction.

LET $f = a_0 x_1^n + \cdots$ be an ordinary algebraical quantic in m variables. Suppose that it is subjected to linear transformations whose coefficients are parameters representing positive residues of a prime number p. The result, $f' = a_0' x_1'^n + \cdots$, will be a quantic whose coefficients will be linear forms in the variables $a_0, a_1, \cdots$ with integral coefficients. Any function φ of the coefficients and variables

$$\varphi = \varphi(a_0, a_1, \cdots; x_1, x_2, \cdots)$$

which possesses the property

$$\varphi(a_0', a_1', \cdots; x_1', x_2', \cdots) \equiv \rho^k \varphi(a_0, a_1, \cdots; x_1, x_2, \cdots) \pmod{p}$$

identically in the a's and x's, ρ being the modulus of the transformation, is called a formal modular covariant of f,* or a formal covariant modulo p of f.

It is the purpose of this paper to develop some invariant processes which produce concomitants of this type; that is, processes which are characteristic of the invariant theory of modular transformations.

§ 1. *Modular Polars.*

It is easy to prove that with reference to m-ary transformations with integral coefficients modulo p the following set of functions is cogredient to the set of variables $x_1, x_2, \cdots, x_m$:

$$x_1^{p^t}, x_2^{p^t}, \cdots, x_m^{p^t} \qquad (t \text{ a positive integer}).$$

In fact if the transformations are

$$(1) \qquad x_i = \lambda_i x_1' + \mu_i x_2' + \cdots + \sigma_i x_m' \quad (i = 1, \cdots, m),$$

we have by the multinomial theorem

$$x_i^{p^t} \equiv \lambda_i^{p^t} x_1'^{p^t} + \mu_i^{p^t} x_2'^{p^t} + \cdots + \sigma_i^{p^t} x_m'^{p^t} \pmod{p}.$$

Hence by Fermat's theorem

$$(2) \qquad x_i^{p^t} \equiv \lambda_i x_1'^{p^t} + \mu_i x_2'^{p^t} + \cdots + \sigma_i x_m'^{p^t} \pmod{p},$$

which proves the statement.

In any formal-modular invariant function $\varphi(x_i)$ we may replace the variables x_i by $x_i + l x_i^{p^t}$ without disturbing the property of invariance. Hence follows the theorem.

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* Hurwitz, *Archiv der Math. und Phys.*, ser. 3, vol. 5 (1903), p. 17. Dickson, Madison Colloquium Lectures, Lecture III, and Lecture IV, p. 68.

THEOREM 1: *The modular polar*

$$(3) \qquad E_m^{(t)} \equiv x_1^{p^t}\frac{\partial}{\partial x_1} + x_2^{p^t}\frac{\partial}{\partial x_2} + \cdots + x_m^{p^t}\frac{\partial}{\partial x_m} \quad (\mathrm{mod}\ p)$$

is an invariant operator.

Any other set of functions $f_1, f_2, \cdots, f_m$ which possesses the property of cogrediency with the variables will furnish a modular polar operator.

For illustration consider the modular polars of the quadratic form in m variables with arbitrary coefficients

$$q_m = \sum_{i,j=1}^{m} a_{ij}x_ix_j \quad (i \leqq j).$$

Operating with $E_m^{(1)}$, we obtain the polars

$$(4) \qquad \begin{aligned} E_m^{(1)}q_m &= \sum_{i<j=1}^{m} a_{ij}(x_ix_j^p + x_jx_i^p) + 2\sum_{i=1}^{m} a_{ii}x_i^{p+1}, \\ \tfrac{1}{2}E_m^{(1)2}q_m &= \sum_{i,j=1}^{m} a_{ij}x_i^px_j^p. \end{aligned}$$

These are both formal-modular covariants of q_m under (1). A direct extension shows that if $n \not\equiv 0 \pmod{p}$, an m-ary form f_n has n covariant polars $E_m^{(t)r}f_n$ $(r = 1, 2, \cdots, n)$ of degree 1 in the coefficients. If $p > n$, none of these polar covariants will vanish modulo p. Thus if $n = 3$, $p > 3$, $m = 2$, we have

$$E_2^{(t)} = x_1^{p^t}\frac{\partial}{\partial x_1} + x_2^{p^t}\frac{\partial}{\partial x_2}, \quad f = a_0x_1^3 + 3a_1x_1^2x_2 + \cdots$$

and

$$\begin{aligned} \tfrac{1}{3}E_2^{(t)}f = {} & a_0x_1^{p^t+2} + 2a_1x_1^{p^t+1}x_2 + a_2x_1^{p^t}x_2^2 + a_1x_1^2x_2^{p^t} \\ & + 2a_2x_1x_2^{p^t+1} + a_3x_3^{p^t+2}, \\ \tfrac{1}{6}E_2^{(t)2}f = {} & a_0x_1^{2p^t+1} + a_1x_1^{2p^t}x_2 + 2a_1x_1^{p^t+1}x_2^{p^t} + 2a_2x_1^{p^t}x_3^{p^t+1} \\ & + a_2x_1x_2^{2p^t} + a_3x_2^{2p^t+1}, \\ \tfrac{1}{6}E_2^{(t)3}f = {} & a_0x_1^{3p^t} + 3a_1x_1^{2p^t}x_2^{p^t} + 3a_2x_1^{p^t}x_2^{2p^t} + a_3x_2^{3p^t}. \end{aligned}$$

The analogy which this polar theory presents when compared with the algebraic polar theory is closer than it might appear to be at first sight. For a little consideration will show that it is immaterial whether we operate directly with

$E_m^{(t)}$, as in the illustrations above, or whether we operate with $E(y) = \left(y\frac{\partial}{\partial x}\right)$ the requisite number of times in succession and then set $y_i = x_i^{p^t}$ $(i = 1, \cdots, m)$. The two results will be identical modulo p. The essential difference between the present theory and the algebraic theory is that an algebraic polar becomes the original polarized form when $(y) = (x)$, whereas in the present theory (y) is expressible in terms of (x) in such a way as to give covariants other than the form itself.

§ 2. *Modular Aronhold Operators.*

Let us suppose that f is a form having $\mu + 1$ coefficients. As previously stated, the coefficients a_i' $(i = 0, \cdots, \mu)$ of the transformed of f under (1) are linear forms with integral coefficients in the variables $a_0, a_1, \cdots$, that is

$$(5)\quad a_i' \equiv \xi_i a_0 + \eta_i a_1 + \cdots + \tau_i a_\mu \pmod{p} \quad (i = 0, 1, \cdots, \mu),$$

where $\xi_i, \eta_i, \cdots$ take all values modulo p which are induced by the linear group (1). Thus (5) form a group induced by (1). Under group (5) the following are cogredient to a_0, $a_1, \cdots, a_\mu$:

$$a_0^{p^t}, a_1^{p^t}, \cdots, a_\mu^{p^t} \quad (t \text{ any positive integer}).$$

Hence (§ 1) the following operator, applied to any concomitant of f, gives a formal concomitant modulo p:

$$\delta_\mu^{(t)} = \left(a^{p^t}\frac{\partial}{\partial a}\right) \equiv a_0^{p^t}\frac{\partial}{\partial a_0} + a_1^{p^t}\frac{\partial}{\partial a_1} + \cdots + a_\mu^{p^t}\frac{\partial}{\partial a_\mu} \pmod{p}.$$

Consider a binary quadratic form

$$f = a_0x_1^2 + 2a_1x_1x_2 + a_2x_2^2.$$

Let $p = 3$. The algebraic concomitants are f and its discriminant D, and we have

$$\left.\begin{aligned} \delta_2^{(1)}f &\equiv a_0^3x_1^2 + 2a_1^3x_1x_2 + a_2^3x_2^2 \\ \delta_2^{(1)}D &\equiv a_0^3a_2 + a_1^4 + a_0a_2^3 \end{aligned}\right\} \pmod{3}.$$

The latter are formal concomitants, modulo 3, of f. The modular Aronhold operator $\delta_\mu^{(t)}$ applied successively to a concomitant of f

$$\varphi(a_0, a_1, \cdots, a_m, x_1, \cdots)$$

gives a series of formal modular concomitants intermediate, in the sense of Boole, to φ and

$$\varphi_t = \varphi(a_0^{p^t}, a_1^{p^t}, \cdots, a_m^{p^t}, x_1, \cdots). \tag{6}$$

If $b_0, b_1, \cdots, b_\mu$ are the coefficients of a second m-ary quantic g of the same order as f, and φ an invariant function of f and g, then

$$\left(a^{p^t}\frac{\partial}{\partial b}\right), \quad \left(b^{p^t}\frac{\partial}{\partial a}\right),$$

applied to φ, give simultaneous formal modular concomitants.

§ 3. *Modular Transvectants.*

We define the modular transvectant of two binary forms

$$f(x) = a_0x_1^m + ma_1x_1^{m-1}x_2 + \cdots, \quad \varphi(x) = b_0x_1^n + nb_1x_1^{n-1}x_2 + \cdots$$

$$m, n \not\equiv 0 \pmod{p},$$

as follows: Operate upon $f(x)\varphi(y)$ with

$$\Omega = \frac{\partial^2}{\partial x_1\partial y_2} - \frac{\partial^2}{\partial x_2\partial y_1}$$

r times, divide by $m!\,n!/(m-r)!\,(n-r)!$ and in the result set $y_i = x_i^{p^t}$ $(i = 1, 2)$. The result, which we abbreviate as $(f, \varphi)_{p^t}^r$, is the rth modular transvectant of f and φ. Thus, if $m = n = 2$,

$$(f, \varphi)_p' \equiv (a_0b_1 - a_1b_0)x_1^{p+1} + (a_0b_2 - a_1b_1)x_1x_2^p + (a_1b_1 - a_2b_0)x_1^px_2$$
$$+ (a_1b_2 - a_2b_1)x_2^{p+1} \pmod{p},$$

$$(f, (f, \varphi)_p')_p' \equiv (a_0a_2 - a_1^2)[b_0x_1^{p^2+1} + b_1(x_1^{p^2}x_2 + x_1x_2^{p^2}) + b_2x_2^{p^2+1}]$$
$$\equiv \tfrac{1}{2}D\cdot E_2^{(2)}\varphi(x) \pmod{p}.$$

I proceed to the problem of finding a canonical formula for $(f, \varphi)_{p^t}^r$, from which several properties can be derived. A well-known form of Gordan's series* gives the expansion of

$$\frac{(m-r)!\,(n-r)!}{m!\,n!}\,\Omega^r f(x)\varphi(y)$$

as a power series in the argument (xy). In the Aronhold symbolical notation ($f(x) = a_x^m$, $\varphi(y) = b_y^n$) this is

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* Grace and Young, Algebra of Invariants, p. 55.

$$(ab)^r a_x^{m-r} b_y^{n-r} = \sum_{i=0}^{m-r} \frac{\binom{m-r}{i}\binom{n-r}{i}}{\binom{m+n-2r-i+1}{i}} (f, \varphi)^{i+r}_{y^{n-i-r}} (xy)^i.$$

The algebraical transvectant $(f, \varphi)^r$ is obtained from this by the change $y = x$, the right hand side reducing to the first term since $(xx) = 0$. The modular transvectant is obtained from this same expansion by the substitution $y = x^{p^t}$. It does not reduce to a single term on the right but becomes a polynomial in the universal formal modular covariant

$$L_t = (x^{p^t} x) = x_1^{p^t} x_2 - x_1 x_2^{p^t}.$$

THEOREM 2:

$$(f, \varphi)^r_{p^t} \equiv \sum_{i=0}^{m-r} N_i \frac{\binom{m-r}{i}\binom{n-r}{i}}{\binom{m+n-2r-i+1}{i}} E_2^{(t)n-i-r} (f, \varphi)^{i+r} L_t^i,$$

where

$$N_i = (-1)^i (m-i-r)!/[m+n-2(i+r)]!$$

In this expansion $E_2^{(t)}$ and L_t are modular and $(f, \varphi)^s$ is algebraic.

We may note that a modular transvectant of a form with itself, of odd index $(f, f)^{2k+1}_{p^t}$, does not vanish identically. But it is reducible in all cases and contains the factor L_t. The transvectants $(f, \varphi)^r_{p^t}$, $(\varphi, f)^r_{p^t}$ are entirely distinct, although they may be called conjugates owing to their symmetrical relationship.

A modular transvectant is a linear combination of modular polars of algebraic transvectants with the universal covariant L_t added to the system. It is known* that L_t is rationally expressible in terms of L_1 and $Q = L_2/L_1$. Moreover I have proved in another paper that Q is a covariant of L_1. Hence one method of procedure in constructing systems of concomitants of a quantic f (modulo p) is to construct the algebraical fundamental system of f and polarize it by the operators $E_2^{(t)}$, $\delta_n^{(t)}$. Then join L_1 to the polar system and form a second system consisting of the simultaneous fundamental system of f and L_1. The forms of the second system which are not found in the polar system are to be added to the polar system. That some forms of the second system will be polars is evident from the fact that, if F is any form whose order is not divisible by p,

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* Dickson, *Transactions*, vol. 12 (1911), p. 75.

$$(L_1, F) = x_1^p \frac{\partial F}{\partial x_1} + x_2^p \frac{\partial F}{\partial x_2} = E_2^{(1)} F \pmod{p}.$$

§ 4. *Concomitants of a Linear Form.*

Let $f = a_0 x_1 + a_1 x_2$; $p = 3$. The algebraical system of f is f itself. Polarizing this, we have

$$C = E_2^{(1)} f = a_0 x_1^3 + a_1 x_2^3, \quad C' = E_2^{(2)} f = a_0 x_1^9 + a_1 x_2^9,$$
$$D = \delta_1^{(1)} f = a_0^3 x_1 + a_1^3 x_2.$$

The modular system of L_1 is

$$L_1 = x_1^3 x_2 - x_1 x_2^3, \quad Q = x_1^6 + x_1^4 x_2^2 + x_1^2 x_2^4 + x_2^6.$$

The simultaneous system of f and L_1 is

$$(L_1, f^r)^r \quad (r = 1, \cdots, 4); \quad (Q, f^s)^s \quad (s = 1, \cdots, 6).$$

Of these, some belong to the polar system and some are reducible; as $(Q, f^2)^2 \equiv fC \pmod{3}$, etc. But

$$A = (L_1, f^4)^4 \equiv a_0^3 a_1 - a_0 a_1^3,$$
$$B = (Q, f^6)^6 \equiv a_0^6 + a_0^4 a_1^2 + a_0^2 a_1^4 + a_1^6,$$
$$E = (Q, f^3)^3 \equiv a_1(a_0^2 - a_1^2) x_1^3 - a_0^3 x_1^2 x_2 + a_1^3 x_1 x_2^2$$
$$+ a_0(a_0^2 - a_1^2) x_2^3.$$

The polars

$$E_2^{(1)} D \equiv f^3, \quad E_2^{(1)} E \equiv D L_1, \quad \delta_1^{(1)} A \equiv 0, \quad \delta_1^{(1)} B \equiv A^2 \pmod{3}$$

are reducible. The polar C' is also reducible. In fact,

$$C' \equiv CQ - fL_1^2 \pmod{3}.$$

The complete set of irreducible concomitants, modulo 3, of the linear form f is

$$A, B, C, D, E, f, L_1, Q.$$

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INVARIANTS, SEMINVARIANTS, AND COVARIANTS OF THE TERNARY AND QUATERNARY QUADRATIC FORM MODULO 2.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, September 8, 1914.)

1. A SIMPLE and complete theory of seminvariants of a binary form modulo p was given in the writer's second lecture at the Madison Colloquium.* A fundamental system of covariants of a ternary quadratic form F modulo 2 was obtained in the fourth lecture. In place of the method employed there (pages 77–79) to obtain the leading coefficient of a covariant of F, we shall now present a simpler method which makes it practicable to treat also the corresponding question for quaternary quadratic forms. The new method is, moreover, in closer accord with the underlying principle of those lectures, viz., to place the burden of the determination of the modular invariants upon the separation of the ground forms into classes of forms equivalent under linear transformation. By making the utmost use of this principle, we shall obtain a simpler solution of the problem for the ternary case and then treat the new quaternary case.

Let the coefficients of the quadratic form

$$q_n = \Sigma b_i x_i^2 + \Sigma c_{ij} x_i x_j \quad (i, j = 1, \cdots, n;\ j > i)$$

be undetermined integers taken modulo 2. In a covariant of order ω of q_n, the coefficient of x_n^ω is called the leader and also a seminvariant. It is invariant with respect to the group G generated by the linear transformations on $x_1, \cdots, x_{n-1}$ and those replacing x_n by $x_n + l$, where l is a linear function of $x_1, \cdots, x_{n-1}$, the coefficients in each case being integers taken modulo 2.

2. For $n = 2$, G is composed of the transformations

$$x_1 = x_1', \quad x_2 = x_2' + tx_1'.$$

Taking $t = b_1$ and applying the transformation to

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* American Mathematical Society Colloquium Lectures, volume IV, New York, 1914; cited later as Lectures.

$$q_2 = b_1x_1^2 + b_2x_2^2 + c_{12}x_1x_2,$$

we get

$$q_2' = \sigma x_1'^2 + b_2x_2'^2 + c_{12}x_1'x_2',$$

where

$$\sigma = b_1(b_2 + c_{12} + 1).$$

Evidently σ, b_2, and c_{12} are seminvariants. Since therefore they completely characterize the classes of forms q_2 under G, they form a fundamental system of seminvariants of q_2. Since c_{12} and

$$J_2 = (b_1 + c_{12} + 1)(b_2 + c_{12} + 1)$$

remain unaltered when b_1 and b_2 are interchanged, they are invariants and, in fact, form a fundamental system of invariants of q_2.

3. For $n = 3$, we have the seminvariants b_3 and

$$P = (c_{13} + 1)(c_{23} + 1),$$

and the invariants (Lectures, pages 69, 74)

$$A = P\,(c_{12} + 1), \quad \Delta = c_{12}c_{13}c_{23} + b_1c_{23} + b_2c_{13} + b_3c_{12},$$

$$J_3 = \beta_1\beta_2\beta_3,$$

of which Δ is the discriminant of q_3, and

$$\beta_1 = b_1 + (c_{12} + 1)(c_{13} + 1), \quad \beta_2 = b_2 + (c_{12} + 1)(c_{23} + 1),$$

$$\beta_3 = b_3 + P.$$

THEOREM. *A fundamental system of seminvariants of q_3 is given by b_3, P, A, Δ, J_3.*

It suffices to prove that they completely characterize the classes of forms q_3 under the group G. We have

$$q_3 = q_2 + lx_3 + b_3x_3^2, \quad l \equiv c_{13}x_1 + c_{23}x_2.$$

(I) $b_3 = P = 0$. Then l is not identically zero and can be transformed within G into x_2. In $q_2 + x_2x_3$ we replace x_3 by $x_3 + c_{12}x_1 + b_2x_2$ and obtain $\Delta x_1^2 + x_2x_3$.

(II) $b_3 = 0$, $P = 1$. Then $l \equiv 0$, $q_3 = q_2$. Thus $A = c_{12} + 1$ and $J_3 = J_2$, which completely characterize the classes q_2 (§ 2).

(III) $b_3 = 1$, $P = 1$. In $q_3 = q_2 + x_3^2$ we replace x_3 by $x_3 + b_1x_1 + b_2x_2$ and obtain $c_{12}x_1x_2 + x_3^2$, where $c_{12} = A + 1$.

(IV) $b_3 = 1$, $P = 0$. As in (I), we may set $l = x_2$. Replacing x_3 by $x_3 + b_1x_1$, we may set also $b_1 = 0$ in q_3. Then $\Delta = c_{12}$, $J_3 = (c_{12} + 1)b_2$. Thus, if $\Delta = 0$, $q_2 = J_3x_2^2$. But, if $\Delta = 1$, we replace x_1 by $x_1 + b_2x_2$ and get $q_2 = x_1x_2$. In either case, the coefficients of the final q_3 are determined by the seminvariants.

COROLLARY. *There are exactly eleven linearly independent seminvariants of* q_3; *they may be taken to be* 1, A, Δ, J_3, AJ_3, b_3, b_3A, $b_3\Delta$, b_3J_3, P, b_3P.

In fact, the number of classes in the four cases was 2, 4, 2, 3, respectively. Hence (Lectures, page 13) there are exactly 11 linearly independent seminvariants. The 11 functions in the corollary can be proved to be linearly independent either in the usual direct manner or more simply by noting that any polynomial in b_3, P, A, Δ, J_3 can be reduced modulo 2 to a linear function of the 11 by means of the relations

$$\Delta A = \Delta J_3 = b_3AJ_3 = 0, \quad PA = P,$$
$$P\Delta = b_3(A + P), \quad PJ_3 = (b_3 + 1)J_3.$$

4. For $n = 4$, the discriminant of q_n is the Pfaffian

$$[1234] = c_{12}c_{34} + c_{13}c_{24} + c_{14}c_{23}.$$

Another evident invariant of q_4 is $A_4 = Au$, where

$$u = (c_{14} + 1)(c_{24} + 1)(c_{34} + 1).$$

We shall employ the abbreviations

$$B_1 = b_1 + (c_{12} + 1)(c_{13} + 1)(c_{14} + 1),$$
$$B_2 = b_2 + (c_{12} + 1)(c_{23} + 1)(c_{24} + 1),$$
$$B_3 = b_3 + (c_{13} + 1)(c_{23} + 1)(c_{34} + 1),$$
$$B_4 = b_4 + u.$$

Then[*] q_4 has the further invariants

$$J_4 = B_1B_2B_3B_4 + b_1b_2b_3b_4[1234],$$

$$\begin{aligned} K = {} & \Sigma b_1b_2c_{34} + \Sigma b_1(c_{23}c_{24} + c_{23}c_{34} + c_{24}c_{34} + c_{23} + c_{24} + c_{34}) \\ & + [1234]\Sigma b_1 + \Sigma c_{12} + \Sigma c_{12}c_{13} + \Sigma c_{12}c_{13}c_{14} \\ & + c_{12}c_{13}c_{24}c_{34} + c_{12}c_{14}c_{23}c_{34} + c_{13}c_{14}c_{23}c_{24} + \Sigma c_{12}c_{13}c_{24}, \end{aligned}$$

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[*] *Proc. Lond. Math. Soc.*, ser. 2, vol. 5 (1907), p. 308.

the subscripts in the final sum being

12 13 24, 12 13 34, 12 14 23, 12 14 34,
13 14 23, 13 14 24, 12 24 34, 12 23 34,
13 23 24, 13 24 34, 14 23 24, 14 23 34.

We have the seminvariants b_4 and u since

$$q_4 = q_3 + \lambda x_4 + b_4 x_4^2, \quad \lambda = \sum_{i=1}^{3} c_{i4} x_i.$$

THEOREM. *A fundamental system of seminvariants of* q_4 *is given by* b_4, u, [1234], A_4, J_4, K.

We prove that they characterize the classes q_4 under G.

(I) $b_4 = u = 0$. Then λ is not identically zero and can be transformed within G into x_3. In $q_3 + x_3x_4$ we replace x_4 by $x_4 + c_{13}x_1 + c_{23}x_2 + b_3x_3$ and get $x_3x_4 + q_2$. Then $c_{12} = [1234]$ and $J_2 = K$ characterize the resulting classes (§ 2).

(II) $b_4 = 0$, $u = 1$. Then $\lambda \equiv 0$, $q_4 = q_3$, and

$$A_4 = A, \quad J_4 = J_3, \quad K = \Delta + A + 1$$

form a fundamental system of invariants of q_3 (§ 3).

(III) $b_4 = u = 1$. If $A_4 = 1$, $q_4 = \Sigma b_i x_i^2 + x_4^2$. Replacing x_4 by $x_4 + \Sigma b_i x_i$, we get x_4^2. If $A_4 = 0$, we may take $c_{12} = 1$, replace x_1 by $x_1 + c_{23}x_3$, x_2 by $x_2 + c_{13}x_3$ and have $c_{13} = c_{23} = 0$. Replacing x_4 as before, we get $x_1x_2 + x_4^2$.

(IV) $b_4 = 1, u = 0$. We may set $\lambda = x_3$ as in (I). Replacing x_4 by $x_4 + c_{13}x_1 + c_{23}x_2$, we have $c_{13} = c_{23} = 0$. Then $[1234] = c_{12}$.

First, let $c_{12} = 0$. Then $K = (b_1 + 1)(b_2 + 1)$, $J_4 = b_3K$. If $K = 1$, then $b_1 = b_2 = 0, b_3 = J_4$, and q_4 is fixed. If $K = 0$, we may set $b_1 = 1$, replace x_1 by $x_1 + b_2x_2 + b_3x_3$ and get $b_1 = 1, b_2 = b_3 = 0$.

Second, let $c_{12} = 1$. Then $J_4 = 0, K = b_1b_2 + b_3$. Replace x_1 by $x_1 + b_2x_3$, x_2 by $x_2 + b_1x_3$, x_4 by $x_4 + b_1x_1 + b_2x_2$. We get

$$x_1x_2 + Kx_3^2 + x_3x_4 + x_4^2.$$

COROLLARY. *There are exactly sixteen linearly independent seminvariants of* q_4; *they may be taken to be the invariants*

$$1, \; k = [1234], \; A_4, \; J_4, \; K, \; A_4J_4, \; kK,$$

the products of b_4 *by the preceding other than* A_4J_4, *and*

$$u,\ b_4u,\ (b_4+1)u\Delta.$$

For, the number of classes in the four cases was 4, 5, 2, 5, respectively. Any polynomial in the six seminvariants given in the theorem can be reduced to a linear function of the sixteen in the corollary by use of*

$$kA_4 = kJ_4 = A_4K = 0,\quad J_4(K+A_4+1) = 0,\quad b_4A_4J_4 = 0,$$

$$uA_4 = A_4,\quad uk = 0,\quad uJ_4 = (b_4+1)J_4,\quad uK = (b_4+1)(u+A_4+u\Delta).$$

5. Consider a covariant of odd order ω of q_4

$$C = Sx_4^{\omega} + S_1x_4^{\omega-1}x_1 + \cdots.$$

Now $x_4 = x_4' + x_1'$ replaces q_4 by q_4' in which

$$(1)\qquad c_{12}' = c_{12} + c_{24},\quad c_{13}' = c_{13} + c_{34},\quad b_1' = b_1 + b_4 + c_{14}.$$

If the latter replaces S_1 by S_1', we have $S_1' = S_1 + S$. Hence S has no term with the factor $c_{12}c_{13}b_1$. Of the functions in the corollary, only J_4, A_4J_4, and b_4J_4 contain $c_{12}c_{13}b_1$, and its coefficients in them are linearly independent. Hence

$$S = I + b_4I_1 + cu + db_4u + e(b_4+1)u\Delta,$$

where I and I_1 are linear combinations of 1, k, A_4, K, kK.

From geometrical considerations (Lectures, page 72),

$$L = (k+1)\{(B_1+1)x_1 + \cdots + (B_4+1)x_4\}$$

is a covariant. The coefficients of x_4^{ω} in iL^{ω} $(i = 1, A_4, K)$ are

$$(k+1)(b_4+1)+u,\ A_4b_4,\ (b_4+1)(Kk+K+A_4+u+u\Delta).$$

After subtracting multiples of iL^{ω} from C we may therefore assume that $c = e = 0$ and that I_1 is free of A_4.

First, let $\omega = 1$. Thus S_1 is derived from S by permuting the subscripts 1 and 4. Then $S_1' = S_1 + S$ gives

$$I = c_{14}I_1 + d(c_{14}+1)\{b_1(c_{24}\alpha + c_{12}c_{34} + c_{34}) + b_4(c_{12}\alpha + c_{13}c_{24} + c_{13})\},$$

where $\alpha = c_{13} + c_{34} + 1$. Let Σ denote the sum of the second member and the function obtained from it by interchanging the subscripts 1 and 2. Thus $\Sigma = 0$. Taking $c_{24} = c_{14}$, we see that $d = 0$. Thus $(c_{14} + c_{24})I_1 = 0$. Apply $x_2 = x_2' + x_1'$, whence

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* The first four from the table of the paper last cited, p. 311.

$$c_{13}' = c_{13} + c_{23}, \quad c_{14}' = c_{14} + c_{24}, \quad b_1' = b_1 + b_2 + c_{12}.$$

Then $c_{14}I_1 = 0$. Hence every $c_{ij}I_1 = 0$, $I_1 = lA_4$. But I_1 is free of A_4. Hence $I_1 = 0$, $I = 0$, $S = 0$.

THEOREM. *Every linear covariant of q_4 is a linear function of L, A_4L, KL.*

Next, let $\omega > 1$. After subtracting from C a constant multiple of $q_4L^{\omega-2}$, whose leader is b_4u, we have $d = 0$ in S. Express S_1 as a polynomial in c_{12}, c_{13}, b_1, and call p the coefficient of their product. The coefficient of $c_{12}c_{13}$ in $S_1' - S_1 = S$, found from (1), is $p(b_4 + c_{14})$, and hence vanishes if $b_4 = c_{14}$; while S itself vanishes if also $c_{24} = c_{34} = 0$. Applying these two conditions to $S = I + b_4I_1$, we find that

$$S = (b_4 + 1)k(n + mK), \qquad n,\ m \text{ constants}.$$

Several tests failed to exclude this leader. Whether or not there are covariants with such a leader S is not discussed here.

In this connection, note the covariant

$$\Sigma c_{ij}(x_i x_j^{2^r} + x_i^{2^r} x_j) \quad (i, j = 1, \cdots, 4;\ i < j),$$

obtained by replacing the variables in the polar of (x) with respect to q_4 by $x_k^{2^r}$ $(k = 1, \cdots, 4)$.

6. By means of the corollary in § 4, and transformation (1), we readily obtain the

THEOREM. *Every quadratic covariant of q_4 is a linear function of L^2, KL^2, Iq_4, where I is an invariant.*

UNIVERSITY OF CHICAGO,
June, 1914.

THE CONVERSE OF THE HEINE-BOREL THEOREM IN A RIESZ DOMAIN.

BY DR. E. W. CHITTENDEN.

(Read before the American Mathematical Society, April 11, 1914.)

IN various generalized forms of the Heine-Borel theorem* for a class $\mathfrak{Q}$ of elements q, the condition that $\mathfrak{Q}$ be extremal* appears as a necessary condition. It is the purpose of the present paper to show that this condition is necessary in any domain for which a Fréchet limit is defined; and furthermore, that if the property extremal be replaced by the property self-compact (a class is self-compact if its every infinite subclass has a limiting element in the class), more general domains may be considered,—notably; those of F. Riesz† for which limiting element is the fundamental relation, and those of T. H. Hildebrandt‡ for which a limit of a sequence need not be unique.§

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* Cf. M. Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo*, vol. 22 (1906), p. 26; and T. H. Hildebrandt, "A contribution to the foundations of Fréchet's calcul fonctionnel," *Amer. Jour. of Mathematics*, vol. 34 (1912), p. 282.

1. *Self-Compact Classes in a Riesz Domain.*

A Riesz domain is formed by a class $\mathfrak{P}$ which admits a relation " limiting element " between single elements and subclasses $\mathfrak{T}$ of $\mathfrak{P}$ with the following properties:

(1) If $\mathfrak{T}$ has a limiting element in $\mathfrak{P}$, $\mathfrak{T}$ contains at least two elements.

(2) If p is a limiting element of $\mathfrak{T}$, p is a limiting element of any class containing $\mathfrak{T}$.

(3) If p is a limiting element of $\mathfrak{T}$, and $\mathfrak{T}$ is divided into classes $\mathfrak{T}_1$, $\mathfrak{T}_2$, then p is a limiting element of $\mathfrak{T}_1$ or $\mathfrak{T}_2$.||

In a Riesz domain an element p is interior ($\mathfrak{R}$)¶ to $\mathfrak{S}$ in case $\mathfrak{S}$ contains p and a single element (distinct from p) of every class $\mathfrak{T}$ of $\mathfrak{R}$ which has p for a limiting element.

A family [$\mathfrak{S}$] of classes $\mathfrak{S}$ is an extended enclosure of a class $\mathfrak{Q}$ if for every element q of $\mathfrak{Q}$ there exists an $\mathfrak{S}_q$ of $\mathfrak{S}$ such that q is interior ($\mathfrak{Q}$) to $\mathfrak{S}$.

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* Cf. Fréchet, loc. cit., p. 7.

† "Stetigkeitsbegriff und abstrakte Mengenlehre," Atti del IV Congresso Internazionale dei Matematiche, Roma, 1908, vol. 2, pp. 18–24.

‡ Loc. cit., p. 241.

§ It is desired to call attention to the fact that it is self-compactness rather than extremality which is essential in many theorems of the calcul fonctionnel of Fréchet, and that by the use of this concept most of the theorems of Fréchet which involve the hypothesis of extremality may be extended to domains for which the limit relation is not subject to the uniqueness condition. This has been done to a large extent for classes which admit a development Δ in a paper presented to the American Mathematical Society in 1913 by Professor Pitcher and Dr. Chittenden, which is not yet published.

|| The three properties just given are equivalent to the ones stated by F. Riesz in his Rome paper already cited. That $\mathfrak{T}$ contains an infinity of elements if it has a limiting element is a consequence of (1) and (3). This is established by successive removals of a single element from $\mathfrak{T}$. There always must remain two at least.

¶ Read: interior to $\mathfrak{S}$ relative to $\mathfrak{R}$. This relativity feature was introduced by T. H. Hildebrandt, loc. cit., p. 268 (10).

A class $\mathfrak{Q}$ has the Heine-Borel property if every family $[\mathfrak{S}]$ which is an extended enclosure of $\mathfrak{Q}$ contains a *finite* subfamily with the same property.

THEOREM I. *If a class $\mathfrak{Q}$ in a Riesz domain has the Heine-Borel property, $\mathfrak{Q}$ is self-compact.*

We have to show that every infinite subclass of $\mathfrak{Q}$ has a limiting element in $\mathfrak{Q}$. Suppose there exists an infinite subclass $\mathfrak{R}$ of $\mathfrak{Q}$ with no limiting element in $\mathfrak{Q}$. From condition (2) it follows that $\mathfrak{R}$ contains a sequence $\{q_n\}$ of distinct elements with no limiting element in $\mathfrak{Q}$. We define $[\mathfrak{S}_n]$, an extended enclosure of $\mathfrak{Q}$, as follows: $\mathfrak{S}_n$ contains all elements of $\mathfrak{Q}$ except the elements $q_{n+1}, q_{n+2}, \cdots$. Every element of $\mathfrak{Q}$ is contained in some $\mathfrak{S}_n$. Let q be any element of $\mathfrak{Q}$ and $\mathfrak{T}$ any subclass of $\mathfrak{Q}$ which has q for a limiting element. Let $\mathfrak{T}_1$ denote the elements of $\mathfrak{T}$ which are in the sequence $\{q_n\}$, $\mathfrak{T}_2$ the remaining elements of $\mathfrak{T}$. By (2) q is not a limiting element of $\mathfrak{T}_1$, and by (3) therefore, q is a limiting element of $\mathfrak{T}_2$. But $\mathfrak{T}_2$ is contained in every $\mathfrak{S}_n$. Hence $[\mathfrak{S}_n]$ is an extended enclosure of $\mathfrak{Q}$, such that no finite subfamily of $[\mathfrak{S}_n]$ contains all the elements of $\mathfrak{Q}$. This is contrary to hypothesis, and the proposition is proved.*

2. *Limiting Element Defined in Terms of Limit. Systems* $(\mathfrak{P}; L)$.

Following T. H. Hildebrandt,† we denote by L (limit) any relation between a sequence of elements and a single element. If a relation L holds between a sequence $\{p_n\}$ and an element p, we say p is a limit of $\{p_n\}$ and write

$$p = L_n p_n.$$

A class $\mathfrak{P}$ and a relation L defined for that class form a system $(\mathfrak{P}; L)$.

The following properties are fundamental in the theory of the relation L:

(1) If a sequence has a limit, the limit is unique.

(2) If a sequence has a limit, its every subsequence, taken in the same order, has the same limit.

(3) The identical sequence $\{p_n\}$, $p_n = p(n)$, has p for a limit. A relation L having a property i ($i = 1, 2, 3, 12, 13, 23, 123$) will be denoted by L^i. An L^{123} is a Fréchet limit.‡

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* If $\mathfrak{Q}$ is finite it is compact in a vacuous sense. Every finite class has the Heine-Borel property.

† Loc. cit., p. 241.

‡ Loc. cit., pp. 6–7.

In a system $(\mathfrak{P}\,;\,L)$, p is a limiting element of class $\mathfrak{T}$ if $\mathfrak{T}$ contains a sequence $\{p_n\}$ of distinct elements having p for a limit.

The following theorem follows readily from this definition of limiting element.

THEOREM II. *In a system* $(\mathfrak{P};\,L^2)$, $\mathfrak{P}$ *is a Riesz domain.*

As an immediate consequence of theorems I and II we have:

THEOREM III. *In a system* $(\mathfrak{P};\,L^2)$, *every class* $\mathfrak{Q}$ *which has the Heine-Borel property is self-compact.*

We may extend the result of Theorem III with the aid of the following lemma.

LEMMA. *In a system* $(\mathfrak{P};\,L^{12})$, *every self-compact class* $\mathfrak{Q}$ *is extremal.*

$\mathfrak{Q}$ is compact by hypothesis. We have to show that $\mathfrak{Q}$ is closed. That is, if p is a limiting element of $\mathfrak{T}$, a subclass of $\mathfrak{Q}$, then p belongs to $\mathfrak{Q}$. If p is a limiting element of $\mathfrak{T}$ then a sequence $\{q_n\}$ of distinct elements of $\mathfrak{T}$ exists which has p for a limit. As an infinite class $\{q_n\}$ has a limiting element q in $\mathfrak{Q}$ because $\mathfrak{Q}$ is self-compact. $\{q_n\}$ therefore contains a subsequence $\{q_{n_k}\}$ of distinct elements with limit q. $\{q_{n_k}\}$ contains a subsequence $\{q_{n_{k_m}}\}$ whose order is the same as that of both $\{q_n\}$ and $\{q_{n_k}\}$. Therefore by L^2, $q = L_m q_{n_{k_m}}$ and $p = L_m q_{n_{k_m}}$. By L^1, q and p are the same and therefore p belongs to $\mathfrak{Q}$; which was to be shown. We have from Theorem III and the lemma

THEOREM IV. *In a system* $(\mathfrak{P}\,;\,L^{12})$, *every class* $\mathfrak{Q}$ *which has the Heine-Borel property is extremal.*

It will be noticed that the character of the result in Theorem IV is determined by the definition of interior given in §1 for Riesz domains. We will show that in a system $(\mathfrak{P}\,;\,L^2)$ this definition is (except for relativity) equivalent to the definition used (implicitly) by Fréchet in the theorem cited at the beginning of this paper. It will follow that Theorem IV includes as a special case the result of Fréchet.

The definition of interior used by Fréchet (except for relativity) is equivalent to the following: q is interior $(\mathfrak{R})$ to $\mathfrak{S}$ if $\mathfrak{S}$ contains q and every sequence $\{r_n\}$ of distinct elements of $\mathfrak{R}$ which has q for a limit is ultimately* contained in $\mathfrak{S}$.

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* In the sense: there exists n_0 such that $n \geqq n_0$ implies r_n is contained in S.

If now p is Fréchet interior to $\mathfrak{S}$ and is a limiting element of $\mathfrak{T}$ a subclass of $\mathfrak{R}$, then, by definition of Fréchet interior, limiting element, and L^2, $\mathfrak{S}$ contains an infinity of elements of $\mathfrak{T}$. Therefore p is interior to $\mathfrak{S}$ in the sense of § 1. Furthermore if p is interior ($\mathfrak{R}$) to $\mathfrak{S}$ in the sense of § 1, then $\mathfrak{S}$ contains an element q (distinct from $\mathfrak{P}$) of every subclass $\mathfrak{T}$ of $\mathfrak{R}$ for which p is a limiting element. Then if $p = L_n r_n$ (distinct) p is a limiting element of the class $[r_n]$. Hence $\mathfrak{S}$ contains r_{n_1} distinct from p. Since p is a limiting element of the class obtained from $[r_n]$ by removing r_{n_1} (L^2) it is evident that at most a finite number of elements of $[r_n]$ are not in $\mathfrak{S}$. Therefore $[r_n]$ is ultimately contained in $\mathfrak{S}$.

T. H. Hildebrandt* has given a definition of interior ($\mathfrak{R}$) which becomes equivalent to the Fréchet interior ($\mathfrak{R}$) for systems ($\mathfrak{P}$; L^{123}). This definition omits the condition that the sequence $\{r_n\}$ consist of distinct elements. If then $p = L_n r_n$ and r_{n_0} is repeated infinitely often, in a system ($\mathfrak{P}$; L^{123}), $r_{n_0} = p$. That r_{n_0} is contained in any class $\mathfrak{S}$ to which p is Fréchet interior ($\mathfrak{R}$) is evident. A restatement of Theorem IV for systems ($\mathfrak{P}$; L^{123}) gives us a generalization of a theorem of Hildebrandt.†

URBANA, ILL.,
October 28, 1914.

COMPLETE EXISTENTIAL THEORY OF SHEFFER'S POSTULATES FOR BOOLEAN ALGEBRAS.

BY PROFESSOR L. L. DINES.

(Read before the American Mathematical Society, December 30, 1913.)

IN a recent number of the *Transactions* Sheffer‡ presented an elegant and concise set of five postulates for Boolean algebras, and proved them mutually consistent and independent. Professor E. H. Moore§ has suggested a further interesting problem in connection with such sets of postulates, namely the determination of all general implicational relations which exist between properties defined either by the postulates themselves or by the negatives of the postulates. A set of postulates are then said to be *completely independent*, if and only if no such implicational relations exist. For example, though Sheffer's postulates are independent in the ordinary sense that no four of them imply a fifth, they are not completely independent, for it can be shown that the negative of the first postulate implies the third, fourth, and fifth.

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* Loc. cit., p. 268 (10).

† Loc. cit., p. 282 (2).

‡ H. M. Sheffer, "A set of five postulates for Boolean algebras with application to logical constants," *Transactions*, vol. 14 (1913), pp. 481–488.

§ E. H. Moore, "Introduction to a form of general analysis," New Haven Mathematical Colloquium, Yale University Press, page 82.

The postulates in question define properties of a system consisting of a class $\mathfrak{K}$ of undefined elements and an undefined binary rule of combination $|$ between elements of $\mathfrak{K}$. Any system $\Sigma(\mathfrak{K}, |)$ of the prescribed type has, with respect to the five properties defined by the postulates, one of the $2^5 = 32$ characters:

$$(1)\qquad \begin{array}{c} (+++++),\quad (-++++),\quad \cdots,\quad (----+), \\ (-----); \end{array}$$

the ith sign of the character being $+$ or $-$ according as Σ has or has not the ith property. The body of $2^5 = 32$ propositions stating for the various characters (1) that there does or does not exist a system having the character in question, constitutes what Professor Moore has called "the complete existential theory" of the five postulates.

§ 1. *Sheffer's Postulates Concerning a System* $\Sigma(\mathfrak{K}, |)$.

The five postulates of Sheffer are:

1. There are at least two elements in $\mathfrak{K}$.

2. Whenever a and b are elements of $\mathfrak{K}$, $a \mid b$ is an element of $\mathfrak{K}$.

Definition. $a' = a \mid a$.

3. Whenever a and the indicated combinations of a are elements of $\mathfrak{K}$,

$$(a')' = a.$$

4. Whenever a, b, and the indicated combinations of a and b are elements of $\mathfrak{K}$,

$$a \mid (b \mid b') = a'.$$

5. Whenever a, b, c, and the indicated combinations of a, b, and c are elements of $\mathfrak{K}$,

$$[a \mid (b \mid c)]' = (b' \mid a) \mid (c' \mid a).$$

In what follows, the fact that a system Σ has the property defined by the ith postulate will be indicated[*] by placing i as a superscript to Σ. The fact that Σ does not have the property defined by the ith postulate will be denoted by placing $-i$ as a superscript to Σ. For example we may express the facts that Σ satisfies postulates 2 and 3, but does not satisfy postulate 4 by Σ^{23-4}.

§ 2. *Complete Existential Theory.*

THEOREM. *For the five postulates 1–5 concerning systems* $\Sigma(\mathfrak{K}, |)$, *the complete existential theory consists of 14 propositions of non-existence, and 18 propositions of existence. In particular the four postulates 2–5 are completely independent. The non-existences are expressed by the proposition*[†]

$$\Sigma^{-1} \supset \Sigma^{345}. \tag{2}$$

To prove proposition (2) we need only note that the hypothesis necessitates either $\mathfrak{K}^{\text{null}}$ (that is that $\mathfrak{K}$ have no elements) or $\mathfrak{K}^{\text{singular}}$ (that is that $\mathfrak{K}$ have only one element). In the former case Σ satisfies postulates 3, 4, 5 vacuously. In the latter case Σ satisfies postulates 3, 4, 5 either vacuously or evidently, according as Σ does not or does satisfy postulate 2.

Proposition (2) renders impossible the existence of systems Σ with the following 14 characters:

(−+−++), (−++−+), (−+++−), (−++−−),
(−+−+−), (−+−−+), (−+−−−),
(−−−++), (−−+−+), (−−++−), (−−+−−),
(−−−+−), (−−−−+), (−−−−−).

We next give examples of systems Σ having each of the other 18 characters as follows: two examples for $\mathfrak{K}^{\text{singular}}$, ten examples for $\mathfrak{K}^{\text{dual}}$, and six examples for $\mathfrak{K}^{\text{triple}}$. In each case the class $\mathfrak{K}$ has the smallest number of elements possible.

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[*] The scheme of notation here described is used by Professor Moore in his General Analysis.

[†] The symbol $\supset$ used in the statement of this proposition is the symbol of logical implication used by Professor Moore and others. In words, proposition (2) may be stated: "If a system Σ does not satisfy postulate 1, then Σ does satisfy postulates 3, 4, and 5."

Examples for $\mathfrak{K}$ singular.

A class $\mathfrak{K}$ with single element m furnishes systems Σ with the following two characters:

$(-++++)$ when $m / m = m$,

$(--+++)$ when $m / m \neq m$.

In the examples for $\mathfrak{K}$ dual, and $\mathfrak{K}$ triple, the operation $\mid$ will be defined by means of tables. For instance if $\mathfrak{K}$ has two elements m and n; and if $m \mid m = n$, $m \mid n = m$, $n \mid m = n$, and $n \mid n$ is not an element of $\mathfrak{K}$, this will be expressed by the table

|	m	n
m	n	m
n	n	—

Examples for $\mathfrak{K}$ dual.

$(+++++)$

|	m	n
m	n	m
n	m	m

$(+--++)$

|	m	n
m	n	—
n	—	n

$(+-+++)$

|	m	n
m	m	—
n	—	n

$(+-+-+)$

|	m	n
m	m	n
n	—	n

$(++-++)$

|	m	n
m	m	m
n	m	m

$(++--+)$

|	m	n
m	n	m
n	m	n

$(+++-+)$

|	m	n
m	m	n
n	n	n

$(+++--)$

|	m	n
m	m	n
n	m	n

(+++ +−)

|	m	n
m	m	m
n	n	n

(+−−−+)

|	m	n
m	n	m
n	—	n

That there are no examples of other characters to be obtained for $\mathfrak{K}$ dual is shown by the two propositions*

(3) $\mathfrak{K}^{\text{dual}} \cdot \Sigma^{-3} : \supset : \Sigma^{5}$

(4) $\mathfrak{K}^{\text{dual}} \cdot \Sigma^{-2} : \supset : \Sigma^{5}$,

both of which follow without difficulty from the definitions involved.

Examples for $\mathfrak{K}$ triple.

(+−++−)

|	l	m	n
l	l	m	n
m	n	n	l
n	m	—	—

(++−−−)

|	l	m	n
l	l	m	n
m	n	l	n
n	m	n	m

(+−+−−)

|	l	m	n
l	l	m	n
m	l	m	n
n	l	—	n

(+−−+−)

|	l	m	n
l	l	m	—
m	l	l	l
n	m	l	m

(++−+−)

|	l	m	n
l	l	m	n
m	l	l	l
n	m	l	m

(+−−−−)

|	l	m	n
l	l	m	—
m	n	l	m
n	—	—	m

That postulates 2–5 are *completely independent* follows from the fact that systems have been exhibited having the $2^4 = 16$ characters (+±±±±).

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* Proposition (3): If the class $\mathfrak{K}$ has exactly two elements and the system Σ does not satisfy postulate 3, then Σ does satisfy postulate 5. The interpretation of (4) is similar.

§ 3. *Concerning a Substitute for Postulate 1.*

The fact that postulate 1 prevents the set of five postulates from being completely independent suggests the desirability of replacing it by another postulate such that the resulting set shall be completely independent. Evidently the new postulate must assume as a minimum number of elements, a greater number than *two* in order that proposition (2) shall not hold.

If postulate 1 be replaced by the statement that $\mathfrak{K}$ shall have at least *three* distinct elements, the difficulty is not overcome, for there still exist no systems having the characters

$$(-+-+-),\quad (-+---),\quad (--++-),\quad (--+--),$$
$$(---+-),\quad (-----),$$

as is shown by propositions (3) and (4).

It seems rather evident that a completely independent set of postulates could be obtained by postulating as the minimum number of elements of $\mathfrak{K}$ a sufficiently great number. How great this minimum must be has not however been determined by the author.

UNIVERSITY OF SASKATCHEWAN.

ON THE CHARACTERISTICS OF THE PRINCIPAL MANUALS OF ELEMENTARY GEOMETRY PUBLISHED IN ITALY IN THE COURSE OF THE LAST FIFTY YEARS.*

BY PROFESSOR MARIO VECCHI.

IN the first half of the nineteenth century, the Elements of Geometry by A. M. Legendre dominated instruction in that subject in Italy. But about 1860, need for a better exposition was felt, and Italian mathematicians, on the initiation of Luigi Cremona, began to turn to the limpid clearness of Euclid. In the year 1867, the Italian government ordered that geometry should be taught on euclidean lines in the classical gymnasia, and in the same year the well-known mathematicians, Enrico Betti and Francesco Brioschi, published an edition of the first books of Euclid for school use,* based in the main on the edition of Viviani. But though this antique edifice is beyond reproach in the harmony of its distinct parts and in the critical and creative power displayed therein, in form and style Euclid's Elements are not altogether in conformity with the tastes of our more modern times. Accordingly, the Minister of Education permitted schools to substitute for Euclid's Elements other works preserving the general euclidean scheme and the rigor of the ancient methods. In consequence of this permission, textbooks appeared, retaining the essential features of Euclid's work, and at the same time enriching the contents with other material more suitable to our epoch. We mention the work of Professors Sannice and D'Ovidio,† wherein for the first time the intuitional concepts, used tacitly by Euclid, are given precise formulation, making the *postulate of motion* the corner stone.

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* Originally published as Supplement No. I of A. R. Kulischer's translation into Russian of J. W. A. Young's Teaching of Mathematics. The English translation here published has been prepared by A. R. Kulischer.

The work by Faifofer‡ is distinguished not only by exceptional clearness of exposition but also by considerable value from the critical point of view. Here, for the first time in elementary works, the theory of equivalence is expressed in the purely geometric form, thus meeting the need pointed out by Duhamel. The work uses considerably fewer postulates than are used by earlier writers, treats proportion by the aid of new, clear, and very simple considerations, and for the first time, indicates clearly what results are independent of the parallel postulate.

A third work, which like the two already named makes systematic use of the postulate of motion, is remarkable for the novelty of its views. I refer to the Elements of Geometry by K. de Paolis.§

Adhering, with Hoüel, to the views of Helmholtz, this author, like Faifofer, postulates the motion of rigid figures, but in a form logically more satisfying, and defines figures as equal when they can be brought to coincidence by motion. De Paolis, exhibiting in this work his deep scientific knowledge and keen critical sense, became the standard bearer of that considerable group of authors who advocated the fusion of plane and solid geometry. He had shown by an actual example that by rejecting the ancient subdivision of geometry useful simplifications can be made in both parts. The plan of fusion was for a long time looked at askance, but in 1900, official permission was given to use it in the schools after the matter of the first three books of Euclid had been studied. Consequently, a number of textbooks have appeared, deviating from the euclidean system, and embodying the results of modern criticism of the principles of geometry, thus bringing to this subject in the schools a greater measure of precision and rigor.

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* Gli elementi d'Euclide, per cura di E. Betti e F. Brioschi, Florence, 1867.

† Sannice e D'Ovidio, Elementi di Geometria, Naples, 1869.

‡ A. Faifofer, Elementi di Geometria, Venice, 1880. French translation by Fr. Talanti, Paris, 1903.

§ K. de Paolis, Elementi di Geometria, Turin, 1884.

The tide reached its height with the appearance of the Elements of Geometry by Veronese.* This well-known scholar, basing his work on a profound analysis of principles, had already set forth the leading thoughts of his geometric system in his work on hyperspace.†

In the above named Elements he gives (provisionally in the form of a school text-book) a rigorous exposition of the logical foundations of geometry. After having excluded the idea of motion, so characteristic of the text-books named above, he enumerates explicitly all the postulates on which his edifice is based, appealing unhesitatingly to intuition in the interests of freer thinking. He starts from a single fundamental notion, that of a point, and from this the other geometric forms are developed without further need of existence postulates. The most characteristic feature of this work is its original theory of equality. As fundamental concept, that of congruence of segments is here taken. The author practices fusion to a certain extent, and treats simultaneously for the straight line, the plane, and space certain special questions, such as the theory of equality, of similitude, and mensuration. In a short period Veronese published a series of editions of his text book, and crowned the whole with a set of text books for the various types of secondary schools, conforming to the curricula of 1900. In each of these works, his fine critical feeling and his great learning are clearly exhibited. To the courses of Veronese the Elements of

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* G. Veronese, Elementi di Geometria, Padua, 1897.

† G. Veronese, Fondamenti di Geometria a piu Dimensioni, Padua. German translation by A. Schepp. Leipzig, 1894.

Geometry by Ingrami* must be adjoined, which independently of the former made analogous contributions to elementary geometry.

In 1900 there was published by F. Enriques† a collection of valuable papers by numerous collaborators on various questions of elementary geometry, dealing in part with the subject matter and in part with problems of teaching. In 1903 F. Enriques and Ugo Amaldi published a work on elementary geometry, in which are combined with admirable harmony, rigorous and scientific exposition of the principles and compliance with pedagogic requirements. The authors, conforming to those previous writers who exclude the principle of motion, adopt the standpoint of Hilbert whose postulates, in their essential points, find a place here. The novelty and rigor of the exposition give value also to the theory of equivalence, which is presented in an unusually complete form. The authors base the correspondence of polygons and prisms on decomposition into equal parts (Duhamel), and for such forms for which this criterion is insufficient (as proved by Rethy, Dehn, and recently Kagan) the concept of "equality of extensions" is introduced, permitting a logically rigorous treatment. In the chapter on proportion, the authors remain on the ground of euclidean definitions, and succeed in reducing the abstract side of the question to a minimum, and in placing the concrete geometric applications naturally in the foreground. From the novel details of the book, there may be selected for mention an exceedingly elegant treatment of the equality of trihedral and polyhedral angles without the use of the parallel postulate. Such are some characteristic features of their excellent work which combines classic clearness of form and perfect limpidity of expression.

We must mention finally the work of De Franchis,‡ written in the manner of Veronese, and making the interesting experiment of introducing a rigorous treatment of the concept of motion into a school text book.

ROME, ITALY.

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* G. Ingrami, Elementi di Geometria, Bologna, 1904.

† F. Enriques, Questioni riguardanti la Geometria elementare, Bologna, 1900. German translation: Die Fragen der Elementargeometrie, Leipzig, 1907.

‡ Naples, 1909.

MATHEMATICAL METHODS IN PHYSICS.

Sur quelques Progrès récents de la Physique mathématique. Par VITO VOLTERRA, Clark University Lectures of 1909, published by Clark University, 1912. 82 pp.

Drei Vorlesungen über neuere Fortschritte der mathematischen Physik. Von VITO VOLTERRA, mit Zusätzen und Ergänzungen des Verfassers. Deutsch von Dr. ERNST LANELA. Sonderabdruck aus dem *Archiv der Mathematik und Physik,* III. Reihe, Band XXII, Heft 2/3. B. G. Teubner. Leipzig.

Leçons sur l'Intégration des Equations différentielles aux Dérivées partielles. Par VITO VOLTERRA. Professées à Stockholm. Nouveau tirage. Paris, Hermann. 1912. 3 + iv + 83 pp.

The first of these books consists of three lectures delivered at Clark University, and afterwards printed by the University.* They have since appeared in the second form in German in the *Archiv der Mathematik und Physik,* (3), 22 (1914), pages 97–182. In the latter form some of the details omitted in the original are supplied. The third book is a reprint of lectures delivered at Stockholm in 1906. There have been added some corrections, and some bibliographical notes. These lectures are striking examples of the intimate relationship between the advance of mathematics and that of physics.

The fundamental notion of the Stockholm lectures is that the theories of the propagation of heat, of hydrodynamics, elasticity, Newtonian forces, and electromagnetism can all be treated from a single point of view—reducing indeed to differential equations of the same general form but of three types, the facts and the processes used following the types. A good supplementary paper to read along with the first part of the lectures on differential equations, containing examples and more detail, is to be found in the *Annales de l'Ecole Normale,* (3), 24 (1907), page 411. The most interesting part of the lectures is the introduction of the notion, due to Professor Volterra, of *function of a line.* In the fifth lecture this notion appears, and is indeed the guide to a generalization of the analytic functions of a complex variable. The particular function of a line used in this lecture is the line integral

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* The volume also contains lectures by Rutherford: "History of the alpha-rays from radio-active substances"; Wood: "The optical properties of metallic vapors"; Barus: "Physical properties of the iron carbides."

$$V = u_0 + \int_A^B (Xdx + Ydy + Zdz),$$

where u_0 is a constant and the line integral extends from a point A to a point B. If we set $X' = (\partial Y/\partial z - \partial Z/\partial y)$, $Y' = (\partial Z/\partial x - \partial X/\partial z)$, $Z' = (\partial X/\partial y - \partial Y/\partial x)$; then, n being the outward normal to the surface enclosed, the line integral in question around a loop will, by Stokes's theorem, be the same as

$$W = \int\int (X' \cos nx + Y' \cos ny + Z' \cos nz)dA.$$

If we let the loop decrease and determine the limit of the ratio of W to the area enclosed, as the vanishing loop approaches a point, the limit in question is nothing else than the projection, on the normal to the surface at the point, of the vector whose components are X', Y', Z', that is, of the *curl* of the vector X, Y, Z. This projection, if found for a point on the path of integration, Professor Volterra calls the derivative of the function of the line V with respect to the surface, and he represents the curl by the symbols

$$X' = \partial V/\partial(yz) \quad Y' = \partial V/\partial(zx) \quad Z' = \partial V/\partial(xy).$$

If now there is a function u whose gradient is (X', Y', Z'), that is, if

$$\partial u/\partial x = X', \quad \partial u/\partial y = Y', \quad \partial u/\partial z = Z',$$

then the convergence of the gradient of u is zero, and u is harmonic since $\nabla^2 u = 0$. When these relations are satisfied we have u and V so related that one may be called the conjugate of the other. This is the generalization referred to of the theory of complex variables. An easy mathematical example is found by setting

$$u = x^2 + y^2 - 2z^2,$$

which is harmonic, and whose gradient is the vector

$$(2x, 2y, -4z).$$

It is conjugate to the function (integral around a loop)

$$V = \int (2zydx - 2zxdy),$$

since the curl of $(2zy, -2zx, 0)$ is $(2x, 2y, -4z)$, the gradient of u. An easy physical example is the field of potential at a fixed origin of a single magnet pole in empty space, as the pole is moved into all possible positions, which gives the function u; and the field of a circuit carrying a unit electric current, as it is moved into all possible positions and shapes, which gives the potential V at the fixed origin. The two functions are conjugate, the first a function of a point, the reciprocal of its distance from the origin; the latter a function of a line, the solid angle it subtends at the origin.

The function of a line of course need not be conjugate to a function u. In case X', Y', Z' is a vector whose convergence is zero, then it may be written as the curl of a vector X, Y, Z, the well-known relation of a vector to its vector potential; and the integral

$$\int\int (X' \cos nx + Y' \cos ny + Z' \cos nz) dA = \int (X dx + Y dy + Z dz)$$

gives a function of a line. It need be remarked that these definitions of function of a line and the differential of a function of a line are generalized somewhat in the calculus of fonctionnelles.

The notion of monogenicity is extended, under the name isogenicity, to space of three dimensions in the following form. Two functions of a complex variable z are monogenic if their differentials at any point have a ratio f which is a function of the point but not of the direction of the differentials. Two functions of a line are isogenic if at each point of the line the vector derivative of each, that is, the curl of the vector which is to be integrated along the line, is parallel to the vector derivative of the other. In vector notation we would write the definition

$$V \nabla B_1 = f V \nabla B_2,$$

where f is independent of the derivative plane as defined above. The idea of function of a line may evidently be extended to functions of surfaces and hyperspaces, and leads into the calcul fonctionnel.

These functions of lines and surfaces, and for the most general case, of hyperspaces, enable one to understand the application of the methods of Jacobi and Hamilton to the problems of the calculus of variations. The simple integral of the ordinary theory may be considered as a function of its limits and of the values of unknown functions at the limits. The extensions to multiple integrals are easily suggested by this view of the procedure, namely, we must consider the integrals to be functions of the lines, or surfaces, or hyperspaces, that bound the space, and of the values of unknown functions on these contours.

The detailed treatment of the partial differential equations is to be found in the last four lectures, and reference to them is necessary to have a clear notion of them.

In the course at Clark University we find the dominant idea again to be the unifying principles of the application of mathematics to mechanics, elasticity, and mathematical physics. The first lecture is devoted to showing the reduction of physical problems to problems in the calculus of variations, the second to the advance in methods in elasticity, and the third to the problem of heredity, which leads to Volterra's integro-differential equations. These three lectures we will examine in some detail.

In the first there is given a reduction of the problem of electrodynamics into terms of the variation of the definite integral

$$P = \int dt \tfrac{1}{2} \cdot \int\int \Sigma(\alpha_{rs} X_r X_s + \beta_{rs} L_r L_s) dA,$$

in which the quantities α, β are quite arbitrary. This variation in terms of purely arbitrary quantities shows us that we can devise an infinity of mechanical explanations or models of electrodynamic phenomena. But this analytic form of the problem enables us to introduce curvilinear coordinates, and thus consider curvilinear spaces. We also are enabled to find the integral invariants, and to apply Volterra's reciprocity theorem which corresponds to Green's theorem, and further to introduce the generalization of the Hamilton-Jacobi methods and the principle corresponding to stationary action and varying action. In order to accomplish this, the functions of lines, surfaces, and hyperspaces have to be used. We thus come into contact with the theory of the inversion of definite integrals, that is, the solution of linear integral equations, and with the study of functions of variables which run over assemblages of curves, of surfaces, etc. We are brought up to the functional calculus, or as it has been called, general analysis. The generalization of the Hamilton-Jacobi method replaces the canonical equations by partial derivatives, and the partial differential equation of Jacobi is replaced by a functional equation.

There is an intimate connection between the partial differential equation, its characteristics, and the theory of waves.* It is simply necessary to consider the variable t as on the same footing as the variables x, y, z. The bearing of this is shown in the closing considerations of the first lecture. For example, in the very simple case of the equation of a vibrating membrane

$$\partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 - \partial^2 u/\partial y^2 = 0$$

we have the partial differential equation corresponding to the vanishing of the variation of the integral

$$V = \int\int\int [(\partial u/\partial t)^2 - (\partial u/\partial x)^2 - (\partial u/\partial y)^2] dx dy dt.$$

The surfaces of discontinuous derivatives and variation of V always equal to zero are then the envelopes of the characteristic cones of the partial differential equation. The general question of waves is therefore considered, and Minkowski's universe noticed. This naturally leads to the Lorentz transformation, and to Poincaré's demonstration that under this transformation the integral whose variation was considered, and which may be called the action, remains invariant.

In the second lecture the development of methods in the theory of elasticity is taken up, particularly those that are connected with the ideas already mentioned. The two great classes of methods of integrating the differential equations of elasticity may be called the method of Green with its extensions, and the method of simple solutions. Green's method further has two divisions, in one of which the conception of Green alone is sufficient to solve the problem, in the other we must add consideration of the characteristics. Green's method starts with Laplace's equation, and depends upon a reciprocity theorem, by means of which from a fundamental solution one is enabled to determine a harmonic function inside a given region when its values on the contour are given. Betti carried the method of Green over into elasticity and extended the reciprocity theorem by the proposition: If two systems of exterior forces determine two systems of displacements in an elastic body, the work done by either in producing the displacement due to the other is the same.

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* Encyclopédie des Math., II_4 1 (II, 22, 8).

The second division of methods along the line started by Green is found in Kirchoff's work on the equation of retarded potential on four variables. He succeeded in solving it by means of a fundamental solution due to Euler, but in order to perceive the real difficulties in the way it is necessary to consider the problem on only three variables. The characteristics enter into the solution radically in this case. The lecturer shows the inherent difference between the cases of three dimensions and four dimensions. Again when the body is not simply connected and the functions can be polydromic the method of Green needs further extensions. The reciprocity theorem enters in as a theorem of the symmetry of a set of coefficients $E_{ij} = E_{ji}$, which occur in the linear equations that give the efforts in terms of the distortions.

The method of simple solutions has been given great power by the development of the theory of integral equations, and the determination of methods of expansion in series of orthogonal functions.

The third lecture introduces the new developments due to Professor Volterra himself and now well-known. These lead to the division of mechanics into the mechanics of no heredity, wherein the state of a system depends only upon the infinitesimally near states preceding, and the mechanics of heredity, in which the state depends upon all the preceding states, thus introducing an action at a time-distance. A simple example is used to illustrate the new problem, the dependence of the angle of torsion of a wire upon the moment of torsion. Instead of Hooke's law

$$\omega = KM,$$

we find it must be expressed by a more elaborate law dependent upon the time

$$\omega = KM(t) + \int_{t_0}^{t} M(t)\varphi(t, \tau)d\tau$$
$$+ \frac{1}{1\cdot 2}\int_{t_0}^{t} d\tau_1 \int_{t_0}^{t} d\tau_2 M(\tau_1)M(\tau_2)\varphi(t, \tau_1, \tau_2) + \cdots.$$

In the case of linear heredity this expression terminates with the second term. There arise now several questions.

(1) What is the significance of the coefficient $\varphi(t, \tau)$?

(2) When φ is known, how may M be found for a given ω?

(3) How is φ determined when it is unknown?

(4) Is it possible to extend these conceptions to the general problem of elasticity?

(5) Is it possible to extend these conceptions to the phenomena of magnetism and electricity?

(6) What phenomena will be amenable to this method of treatment?

For the first three questions it is found that the methods of integral equations are sufficient to furnish the answer. However for the fourth a new type of equation is in evidence, the integro-differential equation. We meet this indeed in the problem of the wire itself when we study its oscillations. The integro-differential equation involves the partial derivatives of the unknown function as well as integrals containing the unknown function. The answer to the fifth question leads also to integro-differential equations for the electromagnetic equations.

To resolve equations of this form a further extension of the method of Green is necessary, and a new reciprocity theorem arises. Fundamental solutions may be found and from these arises the complete solution.

In applying algebra to natural phenomena we have the great advantage that we can postpone to the last moment the specification of the constants that enter the natural problem, and are thus in a much better position to consider the related hypotheses. So too in the functional calculus in all its forms, we are able to postpone the specification of the functions entering the problem until the last moment, and are thus able to avoid vicious hypotheses in the beginning. Without these developments many types of problem become impossible of solution. Further, advances in mathematical physics come from the subsumption under one law of many different classes of phenomena.

In the translation of the Clark University Lectures many details of the transformations in the first lecture are supplied, to the relief of the general reader. Further, the references have been made more precise and have been inserted on the pages as footnotes instead of being collected at the ends of the lectures. Some minor corrections have been made. The section headings have been inserted over the sections.

This edition of these very valuable lectures will be welcomed.

The whole field of functional calculus is a new territory but recently open for settlement, though an adventurous investigator occasionally explored small parts of it in the past century. The important extensions of mathematics have come from the problems of inversion, such as the Galois theory, theory of ideals, differential equations, integral equations, and now the calcul fonctionnel. These developments of Professor Volterra are of the highest importance mathematically aside from all of their physical interest, for the reason that they furnish a very practical path of entry into this new field and occupy a considerable part of the field itself. Fortunately we can follow them more in detail in the two recent courses of his lectures, Leçons sur les fonctions des lignes, and Leçons sur les équations intégrales.

JAMES BYRNIE SHAW.

SHORTER NOTICES.

Mysticism in Mathematics. By HASTINGS BERKELEY. Henry Frowde, Oxford University Press, 1910. vi + 264 pp.

IT may be well to begin by stating what Mr. Berkeley's thesis is not. The kind of mysticism which he thinks he detects in mathematics is not any of the kinds of mysticism that one encounters in the history and philosophy of religion. It is not contended that devotion to mathematics begets or tends to beget in the devotee a sense of an immediate and ineffable union or identification with deity. It is not argued that there is any essential likeness between Euclid and Timæus or between Gauss and Angelus Silesius. What Mr. Berkeley calls mysticism in mathematics is, he says, so called by him because no other name seems to him so appropriate. It appears to be impossible to state with mathematical precision what the thing is. Of course the author is not to be blamed for that. He succeeds in the difficult enterprise of making the matter as clear to the reader as it is to the writer, and that is all that can be reasonably expected.

As nearly as we have been able to make it out, the thesis may be broadly stated to be that, owing to a kind of reaction of language and especially of highly symbolic language upon thought and belief, mathematicians tend to acquire a state of mind in which many purely fictitious things appear to be actual or real things. The thesis, which is very elaborately discussed, is approached from several points of view. The book falls into three parts.

In Part I the method is a priori. The Pythagoreans raised mathematics to the level of a science and the Pythagoreans were "enthusiasts and mystics." The conjunction is suspicious. This and kindred considerations lead Mr. Berkeley to devote about 50 large pages to a very interesting psychological and linguistic dissertation designed to show that we are in constant danger of "the vicious reaction of words upon the process of thought," a reaction having a kind of hallucinatory effect, producing a "tendency to mysticism," a tendency to fancy that all symbols and modes of expression represent actualities instead of representing, as frequently they do, merely conventional or fictitious things.

The argument in Part I is general; it applies to all language. But the more symbolic the language, the more vicious is "the vicious reaction." And Part II (94 pages) is devoted to showing, on the one hand, that mathematicians by constantly talking about imaginaries and infinites in algebra and geometry lose sight of the fictitious and convential character of such things, and, on the other hand, to showing how algebra and geometry may be thoroughly purged of mystical elements.

Part III (113 pages), which deals with metageometry, arrives at the interesting conclusion that it "is an illusion" to suppose "that we can conceive different kinds of distance" or different kinds of space, as euclidean and non-euclidean. The illusion is "engendered by an unrealized reaction of symbolic forms, verbal or algebraic, on the thinker's judgment of his own process of thought."

Stylistically the book is well done. Its message to mathematicians—we mean its caution about treating such fictions as the circular points, for example, as real entities in space—might have been more effectively delivered in a score of pages. If it were shorter, the work would be more useful even for philosophers and other laymen.

CASSIUS J. KEYSER.

Exercices de Géométrie analytique. Par P. AUBERT and G. PAPELIER. Tome Premier. Paris, Libraire Vuibert. 360 pp.

IN this book are given 451 exercises on analytic geometry divided among the following fields: the straight line, 94 exercises; the circle, 69 exercises; parametric equations, 46 exercises; classification of conics, 55 exercises; the general equation of the conic, 31 exercises; centers and diameters of conics, 34 exercises; the general properties of conics, 91 exercises; polar coordinates, 31 exercises. Many of the exercises are accompanied by a figure and about one half of them are solved. In several cases more than one solution is given, so as to illustrate different methods of approaching the same problem. The authors have assumed the student to be familiar with harmonic properties, projective transformations, and similar topics which are given in the United States in a second course in analytics. Homogeneous coordinates are not used, and thus many of the problems are solved by rather tedious methods. In our American colleges the book could be used to good advantage in a second course in analytics in conjunction with such a book as Salmon's Conic Sections. The problems are of too difficult a nature for a first course as given in our institutions.

F. M. MORGAN.

Orders of Infinity. The "Infinitärcalcül" of Paul Du Bois-Reymond. By G. H. HARDY. Cambridge University Press, 1910. viii + 62 pp.

MANY problems of mathematical analysis are adequately treated by consideration of the fact that a function does or does not approach a limit, where we may understand the notion of a limit to include the real infinity, with or without sign, and the complex infinity. Other questions, however, demand more refined investigations,—on the one hand when a limit exists and an estimate is necessary of the rapidity of approach to the limit; on the other hand when no limit exists and some notion of the behavior of the function is still requisite. As early as 1821, Cauchy recognized the usefulness of such considerations in his accurate definition of la plus grande des limites and la plus petite des limites,—today more commonly known as superior and inferior limits.

A great many theorems dealing with the limit-behavior of functions were obtained by Du Bois-Reymond in a series of papers dating from 1871 to 1880. These theorems have been collected, recast according to modern requirements of rigor, and amplified by Hardy in No. 12 of the Cambridge Tracts in Mathematics and Mathematical Physics. The work has been done in a manner admitting of no criticism; the treatment is clear and readable; the proofs are accurate and carefully worded.

Questions of the sort considered are generally reducible to a comparison of the rates of increase of positive real functions, as the positive real independent variable becomes infinite (continuously or over integral values); into this form most of the results are thrown by the author. A considerable part of the tract deals with logarithmico-exponential functions,—those obtained by rational operations, the extraction of roots, and the taking of logarithms and exponentials. Such functions have various properties which render their study easy and important; for instance, they always admit of comparison as regards rate of increase, and relations of comparison may under simple conditions be differentiated and integrated.

The tract contains a sketch of applications to such questions as the convergence of series and integrals, asymptotic formulas, the distribution of prime numbers, and the theory of integral functions of a complex variable.

One is impelled to wonder how much of the fairly extensive notation introduced will be found really desirable in actual use of the results. The notions of inferior and superior limit have won a permanent place; Landau's symbols $O(f)$ and $o(f)$ have in a short time come into such wide use as apparently to insure their retention. It is doubtful whether any further notation will be found necessary.

The tract is printed with the clearness characteristic of the series. The only typographical inaccuracy noted is on page 42, line 12, where $e_{\Delta x}$ should be replaced by $e^{\Delta x}$.

Wallie Abraham Hurwitz.

The Hindu-Arabic Numerals. By David Eugene Smith and Louis Charles Karpinski. Boston, Ginn and Company, 1911. vi + 160 pp.

The origin and development of our present number system is probably given as little thought as anything that is so commonly known and used. For persons who are not interested in things mathematical, as well as for those who are thus interested, The Hindu-Arabic Numerals furnishes information that is by no means current. The authors have given in concise form the history of the characters that every school boy of modern days learns to use in computation. Beginning with an account of the early ideas about these characters, the writers go on with the details of place value, the symbol zero, and the actual physical forms by which the numbers have been represented at various times. In chapter 5 is begun the account of their introduction into Europe. This westward aggression seems to have been very slow at first because the new numerals did not appeal to the traders and were regarded as a sort of novelty by educators. The complete acceptance of this really wonderful system finally took place in the sixteenth and seventeenth centuries.

J. V. McKELVEY.

Vorlesungen über darstellende Geometrie, Band I. Dr. F. v. DALWIGK. Leipzig, Teubner, 1911. xvi + 360 pp.

DESCRIPTIVE geometry is presented in this book in a thoroughly instructive manner. A noticeable balance is preserved between what might be called theoretical and practical. Methods of treatment characteristic of pure geometry are used freely, but at the same time the technical use of the principles involved is given due attention. The frequent comparison of different methods of making a given construction is worthy of mention. Shadow construction is treated with excellent simplicity and completeness throughout the book. The treatment of the projection of the intersection of two surfaces is such as to give the beginner a good introduction to the theory of space curves.

In the latter part of the text, trihedral angles with applications to astronomy, surfaces of revolution, screw surfaces, stereographic projection, parallel perspective, and rectangular coordinates are taken up in some detail. The last section of the book is an introduction to mechanical drawing.

J. V. McKELVEY.

Darstellende Geometrie. By TH. SCHMID, associate professor of geometry at the technical school of Vienna. (Sammlung Schubert, LXV.) Berlin and Leipzig, G. J. Göschen, 1912. vi + 279 pp., and 107 figures.

LIKE the other volumes in the Schubert series, this book is written for beginners, and does not attempt to be an exhaustive treatise. It begins with the ordinary double orthogonal projection, but proceeds rather rapidly, so that only forty pages are required for the usual presentation of rectilinear figures. Considerable emphasis is laid on proportion, and the criteria for determining which of two perspective objects hides the other. A generous number of exercises is added at frequent intervals. At the end of this first part is a good introduction to the theory of shades and shadows, and its application to polyhedra. The second and third chapters treat of the sphere, cylinder, and cone; they comprise two thirds of the volume. The discussion begins with polarity as applied to the circle. Much of the text is divided into two columns, one describing polarity, and the other antipolarity, the product of polarity and a reflection. The first twenty pages are very elementary, but the subject is so presented that problems of considerable complexity are disposed of as easily as the simpler ones. An elementary knowledge of plane analytic geometry is presupposed, but most of the properties of conics that are wanted later are established. In connection with the discussion of curvature use is made of the derivative, and in the rectification of the ellipse a foot-note is provided which expresses the length of arc as an elliptic integral. Mechanical devices for drawing conics are described, and the theory of each explained. A section on the illumination of the sphere is delightful reading. Not only does it present the theory in a clear and concise way, but enlivens it with a discussion of the reasons for including the theory in the discussion; both practical and esthetic grounds are presented that are quite convincing. Different degrees of illumination are expressed quantitatively, and directions given for preparing washes to procure them.

An unusual feature is the extent to which the developments of various space curves on the cylinder and cone are discussed. We meet the conchoid, cardioid, cissoid, serpentine, sine curve, and a number of less well known plane curves as developed from an algebraic intersection.

The detailed discussion of the space quartic curve is also somewhat of a departure, but one that is justified. Without using any other machinery than that provided in the previous part of the text, the appearance of the elliptic, nodal, and cuspidal quartics is found, and some of the properties derived. Here again the theory of shades and shadows is liberally applied.

The last chapter, comprising about one sixth of the volume, is devoted to orthogonal axonometry. Here the presentation is not so successful. Too much detail is given, and the reader does not see readily the advantage of the whole procedure. The elementary projections are given as before, in fact several figures are almost identical with those found in the earlier presentation. After twenty pages, however, the treatment is much more satisfactory. The purpose now appears, and the whole discussion is easier and clearer. A detailed treatment of the determination of the scale in each of three directions is followed by the representation of polyhedra, then by complicated architectural figures. A short discussion of the three round bodies is added, and the results applied to an extension coupling. Only a small number of exercises are given in this last part.

VIRGIL SNYDER.

Die graphische Darstellung. By FELIX AUERBACH. Leipzig and Berlin, B. G. Teubner, 1914. (Aus Natur und Geisteswelt, volume 437.) 97 pp. and 100 figures.

THIS little volume is written for readers without mathematical training. Its purpose is to compare the value of various methods of representation. It compares magnitudes by means of lengths of segments of lines, areas of rectangles with constant bases, areas of squares, circles, sectors, and even volumes of certain solids. Incidentally a mass of statistical information is given in the illustrations. After these preliminary notions follows a full and readable discussion of the idea of a plane curve, including periodic and general trigonometric representations. Finally, automatic tide registers, magnetic fields, seismographs, registering thermometers, solar photographs, and sound waves are briefly discussed.

VIRGIL SNYDER.

Differential- und Integralrechnung. Erster Band: Differentialrechnung. By FR. W. MEYER. Second edition. Berlin and Leipzig, G. J. Göschen, 1912 (Sammlung Schubert, X). xv + 418 pp.

THE first edition of this book was published in 1901. In the present edition a large number of details are altered, but the general development is the same as in the earlier one. Thus, old Art. 12 is now pushed into the first chapter and made Art. 6. The new Art. 15 is much shortened. Maxima and minima of functions of more than one variable are introduced. An appendix is provided which treats of tangents and normals.

VIRGIL SNYDER.

NOTE ON "THE DISCOVERY OF INVERSION."

IN a review* under this title of Professor Bützberger's monograph, Ueber bizentrische Polygone, etc., I referred to a theorem on generalized Steinerian series of circles, which I proved in 1901 in connection with an application of elliptic functions to certain closed linkages.

Since the appearance of this review my attention has been drawn to the fact that A. Hurwitz had previously stated the main part of the theorem in an article, Ueber die Anwendung der elliptischen Funktionen auf Probleme der Geometrie, which appeared in the *Mathematische Annalen*, volume 19, page 65.

As will be noticed, Hurwitz, without proof, merely states the theorem, and his method by which it can be proved is entirely different from that of closed linkages.

This note also applies to a similar statement concerning the same theorem in the "Sprechsaal† für die Encyklopädie der mathematischen Wissenschaften."

ARNOLD EMCH.

CORRECTION.

PROFESSOR Veblen has kindly called my attention to an error in my review of "Monographs on Topics of Modern Mathematics" in the BULLETIN for January, 1914. The criticism against the proof of theorem 19 in the first paper is invalid; the proof as it stands is accurate and complete. Professor Veblen believes also that it is legitimate to use the symbol $\{ABC\}$ in the two ways employed, the language in the one case (as in theorem 7) being essentially an abbreviation of that in the other case (as in assumption II).

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* BULLETIN, vol. 20, p. 413.

† *Jahresbericht d. Deutschen Mathematiker-Vereinigung*, vol. 23, 2. Abt., Heft 3/4, pp. 50–51.

R. D. CARMICHAEL.

NOTES.

AT the November meeting of the Edinburgh Mathematical Society the following papers were presented: By G. N. WATSON: "The convergence of the series in Mathieu's functions"; by D. M. Y. SOMMERVILLE: "Taylor's cubics associated with a triangle in non-euclidean geometry"; by E. L. INCE: "The elliptic cylinder functions of the second kind."

THE annual meeting of the London mathematical society was held November 17, and the following officers elected for the year: president, Sir J. LARMOR; secretaries, J. H. GRACE, T. J. I'A BROMWICH; treasurer, A. E. WESTERN. Professor A. E. H. LOVE delivered an address on "Research in mathematics." Arrangements were made to hold regular monthly meetings during the year. The society now numbers more than three hundred members.

THE summer meeting of the Deutsche Mathematiker-Vereinigung, which was to have been held in September in affiliation with the Hanover meeting of the German naturalists and physicians, was omitted. No provision has been announced for the next meeting.

CONTRARY to the announcements made in the newspapers, nearly all the European universities and technical schools are open as usual. American students provided with passports are welcome.

THE sixty-sixth meeting of the American association for the advancement of science was held in Philadelphia during the week beginning December 28, under the presidency of Dr. C. W. ELIOT.

At the meeting of Section A, the retiring vice-president, Dr. FRANK SCHLESINGER, presented his address on "The object of astronomical and mathematical research." Professor H. S. WHITE is vice-president, and Professor F. R. MOULTON secretary of Section A.

A REGULAR meeting of the National academy of sciences was held at Chicago, December 7, 8, 9; the following papers in mathematics were presented at this meeting: By G. A. BLISS, "A generalization of a theorem of Gauss concerning geodesic triangles"; by L. E. DICKSON, "Recent progress in the theories of modular and formal invariants"; by G. A. MILLER, "The ϕ-subgroup of a group of finite order"; by E. H. MOORE, "On the integration by successive approximations of the ordinary differential equation of the first order in general analysis"; by F. R. MOULTON, "An extension of the process of successive approximation for the solution of differential equations"; by H. S. WHITE, "The synthesis of triad systems Δ_t in t elements, in particular for $t = 31$"; by E. J. WILCZYNSKI, "Conjugate systems of space curves with equal Laplace-Darboux invariants."

THE royal gold medal of the Royal Society of London has been awarded to Professor E. W. BROWN, of Yale University, in recognition of his investigations in mathematical astronomy.

THE technical school of Zürich has conferred its honorary doctorate on Professor H. A. SCHWARZ, of the University of Berlin, on the fiftieth anniversary of his receiving the doctor's degree.

PROFESSOR P. KOEBE, of the University of Leipzig, has accepted a professorship of mathematics at the University of Jena, as successor to Professor J. THOMAE.

DR. W. DEIMEL, of the technical school in Munich, was killed in battle October 23, at the age of 28 years.

AT the University of Illinois, Mr. N. C. GAINES has been appointed instructor in mathematics; Mr. W. H. WILSON and Mr. R. H. MARSHALL have been appointed assistants in mathematics.

ON his seventieth birthday, August 25, 1914, the Teachers' association of South Africa presented to Dr. THOMAS MUIR an illuminated address and a gift of books.

PROFESSOR FREDERICK ANDEREGG, of Oberlin College, has returned from a year's leave of absence.

MR. W. E. EDINGTON, of the University of Colorado, has been appointed professor of mathematics in the University of New Mexico.

AT Pennsylvania State College, Dr. E. R. SMITH has been promoted from an assistant professorship to an associate professorship of mathematics.

AT the University of Texas, Dr. D. F. BARROW has been appointed instructor in applied mathematics, and Mr. F. A. LAMOTTE instructor in pure mathematics.

PROFESSOR R. G. D. RICHARDSON, of Brown University, has been elected a member of the section of mathematics and astronomy of the American academy of arts and sciences.

AT Oberlin College, Miss M. E. WELLS has been appointed acting associate professor of mathematics.

PROFESSOR G. B. GUCCIA, of the University of Palermo, founder of the *Rendiconti del Circolo matematico di Palermo*, died October 29, 1914, at the age of 59 years.

PROFESSOR R. LEHMANN-FILHÉS, of the University of Berlin, died June 5, at the age of 60 years.

THE death is reported of Professor HEINRICH BURKHARDT, of the technical school of Munich, in his fifty-fourth year.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

APOLLONIUS OF PERGA. Treatise on conic sections; edited in modern notation with introduction including an essay on the earlier history of the subject by Sir T. L. Heath. Oxford, Clarendon Press, 1913. 8vo. 424 pp. Cloth. 20s.

BOHANNAN (R. D.). Plane analytical geometry. Columbus, O., R. G. Adams, 1913. $1.60

CARRARA (B.). La s. scrittura i ss. padri e Galileo sopra il moto della terra, ossia il sistema copernicano e la s. scrittura al tempo di Galileo: monografia. Seconda edzione, migliorata ed accresciuta. Milano, G. Palma, di G. Daverio, 1914. 16mo. 170 pp. L. 1.50

CAUER (D.). Neue Anwendungen der Pfeifferschen Methode zur Abschätzung zahlentheoretischer Funktionen. (Diss.) Gottingen, 1914.

COBB (C. W.). Plane analytic geometry. Norwood, Mass., Norwood Press, 1913. $1.00

ECHOLS (W. H.). Evolutionary constructions of the imaginary power of a number and its expression as an exponential function. Charlottesville, 1913.

EMMERICH (A.). Kubische Aufgaben aus der Stereometrie. Nebst Anleitungen zur Losung, Determinationen und den Ergebnissen der Zahlenbeispiele. Weinheim, 1914. 8vo. 8 + 127 pp. M. 2.50

FABARIUS (W.). L. Euler und das Problem Fermats. Cassel, 1914. 8vo. 12 pp. M. 1.20

GANTER (H.) und RUDIO (F.). Die Elemente der analytischen Geometrie. II. Teil: Die analytische Geometrie des Raumes. 5. Auflage. Leipzig, Teubner, 1913. 8vo. Geb. M. 3.00

GROOT (W. F. DE). De theorie der kwadratische vormen van oneindig vele veranderlijken en haar toepassing op de lineaire integraalvergelijkingen. Leiden, 1913. 8vo. 106 pp.

HEATH (T. L.). See APOLLONIUS.

HEILAND (F.). Eine Gruppe von Kurven 3νter Ordnung. Progr. grossherzogl. Reformrealgymnasium, Apolda, 1914.

KREUSER (P.). Ueber das Verhalten der Integrale homogener linearer Differentialgleichungen im Unendlichen. (Diss.) Tübingen, 1914.

LABOUREUR (M.). Cours de calcul algébrique différentiel et intégral. Paris, 1913. Fr. 15.00

LESSER (O.). Die Infinitesimalrechnung im Unterricht der Prima. Zweite verbesserte Auflage. Berlin, Salle, 1912. Geb. M. 2.20

LINDWART (E.). Ueber eine Methode von Laguerre zur Bestimmung des Geschlechts einer ganzen Funktion. (Diss.) Göttingen, 1914.

LOUP (W. M.). Een bijdrage tot de theorie der cyclische oppervlakken en congruenties. (Thesis.) Utrecht, van Boekhoven, 1914. 4to. 80 pp.

MERZ (K.). Parallelflächen und Zentralfläche eines besonderen Ellipsoides und die Steinersche Flache. Beispiele einer quadratischen Transformation. Chur, 1914. Gr. 8vo. 54 pp.

MOZANS (H. J.). Woman in science. New York and London, Appleton, 1914. 8vo. 13+452 pp. $2.50

PAULHAN (F.). Esprits logiques et esprits faux. Seconde édition. Paris, Alcan, 1914. 8vo. 8+388 pp.

PEIRCE (B. O.). A short table of integrals. Abridged edition. Boston, Ginn, 1914. 8vo. 32 pp. Paper. $0.15

PIERPONT (J.). Functions of a complex variable. Boston, Ginn, 1914. 8vo. 14+583 pp. Cloth. $5.00

POND (R. S.). Collineations in space of four dimensions. Lawrence, Kan., University of Kansas, 1913.

RITTER (H.). Beweis des Fermatschen Satzes. Cassel, 1914. 8vo. 12 pp. M. 0.50

ROBERT (E.). Composition des formes quadratiques de quatre et de huit variables indépendantes. Zurich, 1912. 8vo. 56 pp.

RUDIO (F.). See GANTER (H.).

SCHMEHL (C.). Die Algebra und algebraische Analysis nebst einer elementaren Theorie der Determinanten und den Anfangsgrunden der Differentialrechnung in den oberen Klassen von höheren Lehranstalten. 2te Auflage. Giessen, E. Roth, 1914. 316 pp.

SPORER (B.). Ueber geradlinige Transversalen algebraischer Kurven. Wien (*Sitz. Akad.*), 1914. Gr. 8vo. 34 pp.

WEBER (O.). Binormalenflächen mit einer zur Striktionslinie äquidistanten Asymptotenlinie in Beziehung zu Evoluten Ribaucourscher Kurven. (Diss.) Halle, 1914.

WILENSKY (M.). Ueber Besselsche Funktionen. (Diss.) Bern, 1914.

ZEUTHEN (H. G.). Lehrbuch der abzahlenden Methoden der Geometrie. Leipzig, Teubner, 1914. 8vo. 12+394 pp. Geb. M. 17.00

II. ELEMENTARY MATHEMATICS.

BAKER (W. M.) and BOURNE (A. A.). Examples in arithmetic extracted from the Public School Arithmetic. London, Bell, 1914. 8vo. 12+217+52 pp. Cloth. 2s.

BELL (A. H.). A course of geometry theoretical and practical. London, Rivingtons, 1914. 8vo. 6+127 pp. 2s. 6d.

BETZ (W.) and WEBB (H. E.). Plane geometry. Boston, Ginn, 1912. 12mo. 10+332 pp. Cloth. $1.00

BOURNE (A. A.). See BAKER (W. M.).

BUKER (E. F.) and FELTER (W. L.). Arithmetics. 3 books. Books 1–2. New York, Silver, Burdett & Co., 1913. $0.80

CAJORI (F.). School arithmetics; primary book. New York, Macmillan, 1914. $0.35

CAMMAU (P.) et HUOT (J.). Algèbre et géométrie. Classe de première et de seconde A et B. 2me édition. Paris, 1913. Fr. 2.60

Chignell (N. J.). Numerical trigonometry. Oxford, Clarendon Press, 1914. Cr. 8vo. 140 pp. 2s. 6d.

Coates (J. V. H.). A first book of geometry. London, Macmillan, 1912. 8vo. 12+142 pp. 1s. 6d.

Cox (H. C.), Tallant (G. A.) and Umholtz (F. H.). Modern arithmetic, elementary. Chicago, Lyons & Carnahan, 1913. 6+443 pp. $0.60

Dronke (J.) und Lötzbeyer (P.). Mathematische Aufgaben für die Oberstufe hoherer Lehranstalten. Dresden, Ehlemann, 1914. 230 pp.

Durell (F.). Algebra. Book 1. New York, Merrill, 1914. $1.00

Fazzari (G.). Elementi di aritmetica con note storiche e numerose questioni varie per le scuole medie superiori. Parte prima: Numeri interi. Operazioni, divisibilità, numeri primi. 2a edizione. Palermo, Reber, 1911. 12mo. 8+127 pp. L. 1.40

Felter (W. L.). See Buker (E. F.).

Fergusson (J. C.). Fergusson's percentage trigonometry or plane trigonometry reduced to simple arithmetic, with a short description of his percentage compass. London, Longmans, 1914. 8vo. 155 pp. Cloth. 3s. 6d.

Fite (W. B.). Second course in algebra. Boston, Heath, 1914. $0.90

Galdéano (Z. G. de). Congreso de la enseñanza matematica celebrado en Paris del 1 al 4 de Abril. Conferencia dada en la Facultad de Ciencias de Zaragoza el dia 22 Abril. Zaragoza, 1914. 8vo. 39 pp.

Gerlach (A.). Lebensvoller Rechenunterricht. 1ter Teil: Einfuhrung und Unterstufe. Leipzig, Voigtlander, 1914. 215 pp. M. 4.25

Gifford (Mrs. E.). Natural sines to every second of arc and eight places of decimals. Published by Mrs. Gifford, Oaklands, Chard, Somerset, 1914. 6+543 pp. 20s.

Gillespie (F. C.). Worked exercises in elementary geometry. London, Milford, 1914. 3s. 6d.

Grabowski (J.). Benedykta Herberta Arithmetica linealis, Cracoviae, 1577. Krakowie, 1913. 15 pp.

Hansen (H. J. P.) og Rasmussen (H.). Geometriske øvelser. 2 Teile. Kopenhagen, Gyldendalske Boghandel, 1914. 32+70 pp.

Harvey (L. D.). Essentials of arithmetic. Books 1–2. New York, American Book Co., 1914. $0.25+$0.50

Herrick (F. M.). Arithmetical and geometrical calculations by analytical statement and comparative cancellations. Princeton, Ill., Herrick & Horton, 1914. 143 pp. $1.00

Huot (J.). See Cammau (P.).

Jacob (J.). Praktische Methodik des mathematischen Unterrichts. Mit einem Vorwort von E. Mach. Wien, 1914. M. 3.40

Jacquet (E.) et Laclef (A.). Arithmétique du brevet élémentaire et des cours complémentaires. 12me édition. Paris, 1913.

Jahnke (E.). Die Mathematik an Hochschulen für besondere Fachgebiete. (Abhandlungen uber den mathematischen Unterricht in Deutschland, Band IV, Heft 7.) Leipzig, Teubner, 1911. 8vo. 6 + 55 pp. M. 1.80

KNAB (P.). Rechenbuch für höhere und mittlere Lehranstalten. 2te u. 3te Auflage. Freiburg i. Br., Herder, 1914. 240 pp. M. 3.00

KUNDT (F.). Arithmetische Aufgaben mit einem Anhange von Aufgaben aus der Stereometrie für Lyzeen (hohere Madchenschulen) und die unteren Klassen der Studienanstalten. Leipzig, Teubner, 1914. Gr. 8vo. M. 2.00

LACLEF (A.). See JACQUET (E.).

LEIGH (C. W.). See PALMER (C. I.).

LEMOINE (A.). Arithmétique. Théorie, exercices et problèmes. 3e édition. Paris, 1913. Fr. 1.00

——. 160 leçons d'arithmétique. Théorie. 2800 exercices et problèmes. 2me édition. Paris, 1913. Fr. 4.00

LÖTZBEYER (P.). See DRONKE (J.).

MAIR (D. B.). Exercises in mathematics. With answers and hints. London, Macmillan, 1914. Cr. 8vo. 4s. 6d. Without answers and hints. 3s. 6d.

MAXSON (W. S.). Practical self-keyed fundamental number work. Boston, Hammett, 1913. 48 pp. Paper. $0.05

MINET (A.) et PATIN (L.). Cours pratique d'arithmétique, de système métrique et de géométrie. Edition rev. et augm. Paris, 1913.

NUNN (T. P.). The teaching of algebra including trigonometry. London, Longmans, 1914. Cr. 8vo. 7s. 6d.

PALMER (C. I.) and LEIGH (C. W.). Plane trigonometry with 5-place tables. New York, McGraw-Hill, 1914. 8vo. 156+132 pp. Cloth. $1.50

PATIN (L.). See MINET (A.).

PRICE (E. A.). Solutions of the exercises in Godfrey and Siddon's Shorter geometry. Cambridge, University Press, 1914. 8vo. Cloth. 4s. 6d.

RASMUSSEN (H.). See HANSEN (H. J. P.).

SCHLÖMILCH (O.). Fünfstellige logarithmische und trigonometrische Tafeln. 24te Auflage. Braunschweig, Vieweg, 1914. Cloth. M. 1.30

SCHRÖN (L.). Siebenstellige gemeine Logarithmen der Zahlen von 1 bis 108,000 und der Sinus, Cosinus . . . von 10 zu 10 Sekunden, nebst einer Interpolationstafel zur Berechnung der Proportionaltafel. 28te revidierte stereotyp-Ausgabe. Braunschweig, Vieweg, 1914. Half calf. M. 7.30

SLICHTER (C. S.). Elementary mathematical analysis. A textbook for first year college students. New York, McGraw-Hill, 1914. 8vo. 490 pp. $2.50

SMITH (T. C.). Problem papers in arithmetic for preparatory schools. London, Bell, 1914. Cr. 8vo. 1s. 6d.

SMOLEY (C. K.). Tables. 8th edition. New York, McGraw-Hill, 1914. 21+330 pp. $3.50

SPRAGUE (C. E.). Accountancy of investment, with which are incorporated logarithms to 12 places and their use in interest calculations. New York, Ronald, 1914. Half leather. $5.00

TALLANT (G. A.). See COX (H. C.).

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L R RY

THE EIGHTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

The eighth regular meeting of the Southwestern Section of the Society was held at the University of Nebraska, Lincoln, Nebraska, on Saturday, November 28, 1914. About twenty-five persons attended the sessions, including the following sixteen members of the Society:

Professor Henry Blumberg, Professor W. C. Brenke, Professor E. W. Davis, Professor H. C. Feemster, Professor A. B. Frizell, Dr. Elizabeth B. Grennan, Professor E. R. Hedrick, Professor Louis Ingold, Professor O. D. Kellogg, Dr. S. Lefschetz, Professor U. G. Mitchell, Professor W. H. Roever, Professor Oscar Schmiedel, Professor E. B. Stouffer, Professor C. E. Stromquist, Professor W. D. A. Westfall.

The morning session opened at 10 A.M. and the afternoon session at 2 P.M. Professor Davis presided. It was decided to hold the next meeting of the Section at Washington University in St. Louis on Saturday, November 27, 1915. The following program committee was appointed: Professor W. H. Roever (chairman), Professor J. N. Van der Vries, Professor O. D. Kellogg (secretary). Attending members were given a smoker on Friday evening at the Commercial Club, and a lunch at the same place on Saturday.

The following papers were presented at this meeting:

(1) Professor P. J. Daniell: "The end correction for an open cylinder."

(2) Professor Oscar Schmiedel: "Multiplication and division by variable operators."

(3) Professor Louis Ingold: "Note on surfaces with a single first normal."

(4) Dr. S. Lefschetz: "Double integrals of Picard for an algebraic variety."

(5) Dr. S. Lefschetz: "The equation of Picard-Fuchs for an algebraic surface with arbitrary singularities."

(6) Professor Henry Blumberg: "On the factorization of certain polynomials and certain linear homogeneous differential expressions."

(7) Professor A. B. Frizell: "The well-ordering of infinite permutations."

(8) Dr. H. M. SHEFFER: "Deductive systems and postulate theory; I. finite case" (preliminary communication).

(9) Professor E. R. HEDRICK and Miss E. A. WEEKS: "On a definition of discrete oscillation."

(10) Professor W. C. BRENKE: "Convergence of an infinite determinant."

(11) Professor E. W. DAVIS: "The distance between two complex points in ordinary space."

In the absence of the authors, Professor Daniell's paper was presented by Professor Kellogg, and Dr. Sheffer's by Professor Hedrick. Abstracts of the papers follow.

1. When a current flows out of a cylindrical tube of radius a into a large bath, a certain correction ka, where k is some fraction, has to be added to the length of the tube in calculating the resistance. Rayleigh in his Theory of Sound has shown that $.785 < k < .849$ and, in the appendix, that $k < .8242$. Certain integrals containing Bessel functions are evaluated by means of asymptotic series, and then the approximate solution of an infinite number of linear equations in an infinite number of variables, obtained from the usual minimum condition, yields the value of k to any required accuracy. Professor Daniell has found by this means that $.8225 < k < .8232$.

2. In this paper, Professor Schmiedel considers a method of multiplication and division by variable multipliers and divisors for the purpose of extending certain tables which are functions of two arguments to negative integral values of both arguments, and makes application of the method to classes of tables of which the ordinary binomial coefficients are a special case.

3. In a previous paper Professor Ingold has shown that for the proof of the formulas of Gauss and Codazzi the usual requirement, that the n-dimensional space under consideration lie in a linear space of $n + 1$ dimensions, can be replaced by the assumption that the given space have a *single first normal* (i. e., a normal vector which is linearly expressible in terms of the tangent vectors and their *first* derivatives). No example, however, was given of a locus satisfying the last requirement without satisfying the first. In this note such examples are furnished for n-dimensional loci, and it is shown that the

only two-dimensional loci of this character are surfaces generated by the tangents to a curve in space of four or more dimensions.

4. In this paper Dr. Lefschetz deals with integrals of the type $\int\int [A(x, y, z, t)dydz + Bdzdx + Cdxdy]$ for which

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = 0.$$

A, B, C are rational on the variety V having for equation $f(x, y, z, t) = 0$. It is shown that, as could be expected, the number of integrals of this type which are of the second kind proper is $R_2 - \rho$, R_2 being the two-dimensional connectivity of V, ρ the number of its simple integrals of the third kind. The following interesting result is obtained: if A, B, C are of the type

$$\frac{\partial R}{\partial y} + \frac{\partial R'}{\partial z}, \quad \frac{\partial S}{\partial z} + \frac{\partial S'}{\partial x}, \quad \frac{\partial T}{\partial x} + \frac{\partial T'}{\partial y},$$

then there exist three *rational functions* of (x, y, z, t), say α, β, γ such that

$$A = \frac{\partial \beta}{\partial z} - \frac{\partial \gamma}{\partial y}, \quad B = \frac{\partial \gamma}{\partial x} - \frac{\partial \alpha}{\partial z}, \quad C = \frac{\partial \alpha}{\partial y} - \frac{\partial \beta}{\partial x}.$$

5. Dr. Lefschetz's second paper appears in full in this number of the BULLETIN.

6. Let

$$P = p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_n(x)y$$

be a linear homogeneous differential expression belonging to the domain $\mathfrak{R}$ of rational functions (i. e., whose coefficients are elements of $\mathfrak{R}$). Denote by d_0 $d_1, \cdots, d_n$ the degrees of $p_0(x)$, $p_1(x)$, $\cdots$, $p_n(x)$ respectively. Professor Blumberg proves the following theorem:

If $d_n - d_0 > 0$ and $d_\nu - d_0 \leqq 0$ $(\nu \neq n)$, P is not expressible as a symbolic product RS, where R and S are linear homogeneous differential expressions belonging to $\mathfrak{R}$ and of the sth and rth order respectively, unless $d_n - d_0 \equiv 0 \text{ mod. } (n/[r, s])$.

Here $[r, s]$ = G.C.D. of r and s. As a particular case we have the theorem

If $d_n - d_0 > 0,\ d_\nu - d_0 \leqq 0\ (\nu \neq n)$ and $[d_n - d_0, n] = 1$, P is irreducible.

The proofs are algebraic in character, no integrals being used. Moreover corresponding theorems hold for the factorization of the polynomial P in y

$$P = p_0(x_1, x_2, \cdots, x_k)y^n + \cdots + p_n(x_1, x_2, \cdots, x_k),$$

where the coefficients are rational functions of their arguments. The application of the results to the integrals of the corresponding differential equations and to the algebraic functions defined by $P = 0$ is evident. Generalizations are given.

7. By an extension of the process developed in a former paper (Chicago, April, 1914) Professor Frizell shows that the possibility of well-ordering the permutations of an ω-series would carry with it the existence of an ω-series of ordinal types—of which the type of this set of permutations is the lowest—whereof no one can be put into one-to-one correspondence with its predecessor.

8. By means of various "transformations of relational coordinates," and corresponding definitions of equivalence of systems, Dr. Sheffer develops a general postulate theory for deductive systems involving a finite number of elements.

9. In generalizations of the theory of functions of real variables, it seems desirable to suppress as far as possible the notion of length. Such other concepts as that of oscillation are therefore difficult of extension. In this paper, Professor Hedrick and Miss Weeks define a concept which will be called discrete oscillation, and they show that many of the usual theorems concerning oscillation remain true for functions of a real variable when this new concept is substituted for the ordinary oscillation. For functions of a real variable the discrete oscillation is defined in terms of any monoton decreasing set of numbers $a_1, a_2, a_3, \cdots, a_n, \cdots$ which approach zero, and is by definition equal to a_i if the ordinary oscillation ω satisfies the inequality $a_{i-1} > \omega \geqq a_i$. Among theorems whose validity is proven are the theorem that the points at

which the oscillation is not less than a given positive number form a closed set; the Du Bois-Reymond theorems on integration; the Sierpinski theorem; and others. The new concept lends itself readily to broad generalizations, and its simplicity suggests the possibility of advantageous use even in the usual theory.

10. Let a_{ik} be the general element of the infinite determinant D and assume the convergence of $\Sigma|a_{ik}|$. By comparison with an infinite product Professor Brenke obtains the following results, of which (*d*) is a well-known theorem, from which also (*a*) might be derived: (*a*) D converges absolutely to the value 0; (*b*) if the elements of any number of rows or columns of D are replaced by quantities less in absolute value than a positive constant, the new determinant converges absolutely to 0; (*c*) if all the elements a_{ik}, $i > k$, are replaced by quantities less in absolute value than 1, the new determinant converges absolutely; (*d*) von Koch's "normal determinant" converges absolutely; (*e*) a normal determinant remains absolutely convergent if elements a_{ik} are replaced as in (*c*).

11. Professor Davis shows that if the difference between two complex vectors in space is $\delta_1 + \sqrt{-1}\,\delta_2$ and if k is $UV\delta_1\delta_2$, then the square of the distance between the complex vectors is $e^{\sqrt{-1}\theta}T(\delta_1 + k\delta_2)T(\delta_1 - k\delta_2)$ where θ is the angle between $\delta_1 + k\delta_2$ and $\delta_1 - k\delta_2$. This is an extension to space of a formula of Laguerre.

O. D. Kellogg,
Secretary of the Section.

NOTE ON THE POTENTIAL AND THE ANTI-POTENTIAL GROUP OF A GIVEN GROUP.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society at Chicago, December 29, 1914.)

§ 1. *Introduction.*

With every regular substitution group[*] there may be associated a conjugate substitution group on the same letters such that each of these two groups is composed of all the possible substitutions on these letters which are commutative with every substitution of the other group. This fundamental theorem was proved by C. Jordan in his doctor's dissertation* and the given groups are called conjoints in Jordan's Traité des substitutions, 1870, page 58. Somewhat later G. Frattini made a more detailed study of the properties of these two related groups, and called them the *potential* and the *antipotential group* respectively.†

Some of the results obtained by Frattini seem to have received less attention than they deserve. This may be partly due to the facts that his developments are not always clear and that his erroneous statements tend to create a feeling of distrust on the part of the reader. One of these statements appears in the form of a theorem on page 168 of the article to which we have referred, and is so obviously incorrect that it appears curious to find that Netto repeated it in his review of the article in the *Jahrbuch über die Fortschritte der Mathematik,* volume 15, page 113.

This theorem affirms that if we form successive maximal subgroups, starting with a group G and ending with identity, then the series of integers obtained by dividing the order of each of these groups by the order of the one which immediately follows it in the given series is an invariant of G, except possibly as regards the order in which the integers occur. The fact that this theorem is false can be readily seen by means of such a well known group as the icosahedral group. Equally obvious misstatements appear in the continuation of the same article, published in volume 18 of the same journal, but these errors also pass without mention by the reviewer in the *Fortschritte.*

In the present note we aim to prove in a simple manner a few of the results obtained by Frattini, and to extend some of his developments along certain lines. For brevity, the potential and the antipotential group will be represented by G and G' respectively. We shall first establish the following elementary theorem:

The conjoint of every transitive constituent of a subgroup of G is a transitive constituent of a subgroup of G', and vice versa.

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* C. Jordan, Thèse présentées a la Faculté des Sciences de Paris, 1860, p. 39.

† G. Frattini, *Atti della R. Accademia dei Lincei. Memorie,* vol. 14 (1883), p. 144.

As a special case of this theorem it follows that every abelian transitive constituent of a subgroup of G is a transitive constituent of a subgroup of G', and vice versa. As a special case of this special case it results that *every cycle of every substitution of one of two conjoints is also a cycle of some substitution of the other.*

To prove the theorem in question let K represent the conjoint of a transitive constituent of a subgroup of order k contained in G. This subgroup is clearly composed of all the substitutions of G which transform K into itself, and hence K has g/k conjugates under G, g being the order of G. These conjugates are on distinct sets of letters. The g/k conjugates of each substitution of K may therefore be represented in the form of a single substitution which is invariant under G. Hence it is possible to establish a (1, 1) correspondence between the conjugates of K so as to obtain a subgroup of order k in G', and this subgroup involves as one of its transitive constituents the conjoint of an arbitrary transitive constituent of the arbitrary subgroup of G. Hence the theorem in question is established.

§ 2. *Conjugate Operators and Conjugate Subgroups.*

The above considerations give rise to a method for finding all the subgroups of G which are conjugate with any one H of its subgroups. If H' is any one of the subgroups of G' which has for one of its transitive constituents the conjoint of a transitive constituent of H, then it results that a necessary and sufficient condition that a given subgroup of G is conjugate with H is that it involves at least one transitive constituent which is the conjoint of some transitive constituent of H'. That is,

A necessary and sufficient condition that two subgroups of G are conjugate under G is that it is possible to select one transitive constituent from each of these two subgroups such that the conjoints of these two transitive constituents are transitive constituents of the same subgroup of G'.

Each transitive constituent of a subgroup of G is completely determined by the letters of this constituent. That is, if a subgroup of G has a transitive constituent involving a certain set of letters, no other subgroup of G has a transitive constituent on exactly the same set of letters. Hence the theorem stated at the end of the preceding paragraph may also be stated as follows:

A necessary and sufficient condition that two subgroups of G are conjugate is that it is possible to find a transitive constituent in each of these two subgroups such that these two transitive constituents involve separately the same letters as two transitive constituents of some subgroup of G'.

As a special case of this theorem it results that a necessary and sufficient condition that a subgroup H of G is invariant is that G' involves a subgroup H' all of whose transitive constituents are the conjoints of the transitive constituents of H. As a special case of this special case it may be observed that

A necessary and sufficient condition that a subgroup of G is invariant is that the totality of the cycles in its substitutions is the same as that of a subgroup of G'.

If we select two complete sets of conjugate subgroups belonging to G and G' respectively, then either all the conjoints of the transitive constituents of the subgroups of the first set are transitive constituents of subgroups of the second set, or none of these conjoints are among the transitive constituents of the subgroups of the second set. In particular, two complete sets of conjugate substitutions belonging to G and G' respectively either have all their cycles in common, or they have no transitive constituent in common. A necessary and sufficient condition that a transitive constituent of a subgroup of G is also a transitive constituent of a subgroup of G' is that this subgroup is abelian. Hence it is possible to find all the conjugates of a subgroup by noting the transitive constituents of subgroups in the potential and antipotential representations.

§ 3. *Methods of Constructing the Potential and the Antipotential Group.*

To construct the potential and the antipotential group of a given abstract group A, Frattini proceeded practically as follows:[*] Let $1, s_2, s_3, \cdots, s_g$ represent the distinct operators of A and construct the following two rectangular arrays:

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[*] G. Frattini, loc. cit.

$$\begin{array}{ccccc} 1 & s_2 & s_3 & \cdots & s_g \\ s_2 & s_2^2 & s_3s_2 & \cdots & s_gs_2 \\ s_3 & s_2s_3 & s_3^2 & \cdots & s_gs_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_g & s_2s_g & s_3s_g & \cdots & s_g^2 \end{array} \qquad \begin{array}{ccccc} 1 & s_2 & s_3 & \cdots & s_g \\ s_2^{-1} & 1 & s_2^{-1}s_3 & \cdots & s_2^{-1}s_g \\ s_3^{-1} & s_3^{-1}s_2 & 1 & \cdots & s_3^{-1}s_g \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_g^{-1} & s_g^{-1}s_2 & s_g^{-1}s_3 & \cdots & 1 \end{array}$$

The substitutions which represent the permutations of the operators in the first array, if each row is compared with the first row, constitute the potential group G of A, while those of the second array constitute the antipotential group G' of A. Both of these groups are simply isomorphic with A and hence they are also simply isomorphic with each other.

Before the publication of this paper by Frattini it was customary to obtain the regular representation of a given substitution group by means of a function which changes its value for every substitution of the group, but during the same year in which Frattini's article appeared there appeared in the *Mathematische Annalen*, volume 22, an important article by Dyck entitled "Gruppentheoretische Studien II," in which the regular representation is derived in the same manner. In fact, this paper by Dyck and the paper by Frattini have many other things in common, but the more lucid presentation of these subjects in the former article has caused an undue neglect of the latter.

A simple isomorphism between G and G' may be obtained by letting the substitutions obtained from each pair of corresponding rows in the given arrays correspond to each other. If the inverses of operators in the second array are replaced by the operators themselves, a simple isomorphism between G and G' results. Moreover, it is clear that in the simple isomorphism between G and G', which results directly from the given arrays, the first cycle of each substitution of G is the inverse of the first cycle of the corresponding substitution of G'.

This fact, combined with the fact that if the first two letters of each pair of corresponding substitutions of two simply isomorphic regular groups are the same then the corresponding substitutions are identical, leads to a very simple method for deriving G from G' or vice versa. This method is embodied in the following theorem:

The substitution of order 2 which transforms into their inverses each of the cycles of G or G' which involve a given letter, and leaves this letter fixed, transforms G and G' into each other.

Hence it is very easy to find either one of two conjoints when the other is given.

A substitution of order 2 which transforms the potential and the antipotential group into each other may be found from the fundamental group itself without the use of the regular representation. In fact, if $1, s_2, s_3, \cdots, s_g$ are the operators of the fundamental group and if we associate g different letters with these operators, then cycles of these letters correspond to the various distinct powers of these operators. The substitution of order 2 in question may be obtained by finding the substitution which transforms each of these cycles whose order exceeds two into its inverse and omits the letter which corresponds to identity.

To illustrate this method, consider the group of order 8 on four letters, and represent its substitutions as follows:

$$1 = A,\quad ac \cdot bd = B,\quad ac = C,\quad bd = D,$$
$$abcd = E,\quad adcb = F,\quad ab \cdot cd = G,\quad ad \cdot bc = H.$$

The only cycle on the capital letters whose order exceeds 2 is $EBFA$. Hence EF transforms the potential and the antipotential group of the octic group into each other. That is, if the octic group is represented in the regular form on the given capital letters in accord with the usual process, then the transposition EF will transform this regular group into its conjoint and vice versa. From the preceding developments there follows the theorem:

If each operator of any given group A is represented by a distinct letter, then the substitution which transforms into their inverses the cycles representing the various distinct powers of all those operators whose orders exceed 2, and omits the letter corresponding to identity, transforms the conjoints of A into each other, if these conjoints are written on these letters in the usual way.

When the group A is an abelian group it is identical with its conjoint and hence it is transformed into itself by the substitution noted in the preceding theorem. In fact, in this case the substitution in question merely transforms into its inverse every operator of A and hence it is contained in the holomorph of A. In general, this substitution transforms the holomorph of G into itself; and when G is non-abelian this substitution and the holomorph of G generate the double holomorph of G. In regard to the given simple isomorphism between G and G' it may be added that if we let every operator of any group correspond to its inverse we get a simple isomorphism only when the group is abelian; but if we multiply in one of these two corresponding groups on the right and in the other on the left there always results a certain kind of simple isomorphism. Moreover, it is easy to verify that if this kind of simple isomorphism can be established between two groups these groups must be actually simply isomorphic.

As regards the theorems relating to conjugate subgroups and invariant subgroups which were developed in § 2, it should perhaps be emphasized that these theorems enable us to determine at a glance all the conjugates of a given subgroup, or of a given operator, provided the potential and the antipotential representation of the group are before us. Similarly they enable us to see at a glance what substitutions and what subgroups are invariant. On the other hand, it should be observed that the representation of a given group in the potential and the antipotential form is laborious, so that these theorems appear to be of more theoretical than practical interest.

THE EQUATION OF PICARD-FUCHS FOR AN ALGEBRAIC SURFACE WITH ARBITRARY SINGULARITIES.

BY DR. S. LEFSCHETZ.

(Read before the Southwestern Section of the American Mathematical Society, November 28, 1914.)

1. LET $F_m(x, y, z) = 0$ be the equation of an algebraic surface of order m with *arbitrary singularities*, the axes having an arbitrary position, and let

$$\int \frac{P(x, y, z)dx}{F_z'}$$

be an abelian integral of the second kind attached to the curve C_y in (xz), having for equation $F_n = 0$. If p is the genus of C_y, then the periods of this integral, considered as functions of y, satisfy a linear homogeneous differential equation of order $2p$: $E_{2p} = 0$, called the equation of Picard-Fuchs. Introduced by Fuchs, a complete study of it was made by E. Picard* for the case where the surface has only ordinary singularities, that is, a double curve with a finite number of points triple for both the curve and the surface. As it is as yet unproved that any variety of more than two dimensions can be birationally transformed into one having only "ordinary singularities," it seems important, for the complete generality of investigations bearing upon integrals belonging to varieties, first to extend the most important results obtained by Picard to surfaces with arbitrary singularities, and this is the object of the present note.

2. Let $\omega_1, \cdots, \omega_{2p}$ be a fundamental set of periods for the integral in question, ω any other period. When y describes a closed path, ω may be changed into ω', and we have

$$\frac{d^k\omega'}{dy^k} = \sum_{i=1}^{2p} m_i \frac{d^k\omega_i}{dy^k},$$

the integers m being independent of k; hence the differential equation

$$E_{2p}(y) = \left\| \frac{d^k\omega}{dy^k}, \quad \frac{d^k\omega_1}{dy^k}, \quad \cdots, \quad \frac{d^k\omega_{2p}}{dy^k} \right\| = 0$$
$$(k = 1, 2, \cdots, 2p)$$

remains unchanged, and if it is written in the form

$$\sum_0^{2p} p_i \frac{d^{2p-i}\omega}{dy^{2p-i}} = 0 \quad (p_1 = 1),$$

the p's, being uniform everywhere in the y plane and regular at infinity, are rational in y.

3. Let $a_1, a_2, \cdots$ be the winding points of the Riemann surface C_y, Q_i a cross-cut traced in the x-plane from a fixed point A around a_i and back to A. Any period is of the type

$$\int \frac{P(x, y, z)dx}{F_z'},$$

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* Picard-Simart, Traité des Fonctions algébriques de deux Variables, vol. 1. p. 94.

the sum being extended to a system of cross-cuts $\Sigma \pm Q_i$ forming a closed path on the Riemann surface. P being taken as a polynomial, this period may cease to be holomorphic where and only where two or more of the winding points coincide. Let (a, b, c) be such a place. When y describes a closed path around b, the a's undergo a substitution, product of several cyclic substitutions having no common letters. Let

$$(a_1, \cdots, a_h)$$

be one of these cycles. For $y = b$, we have

$$a_1 = a_2 = \cdots = a.$$

When y describes a closed circle of small radius around b the a's form the vertices of a regular polygon of h sides having a for center. The axes being arbitrary, the curve C_b is not reducible; for otherwise, that is, if all plane sections of F through (a, b, c) were reducible, the surface would be a cone of vertex (a, b, c) (theorem of Castelnuovo) — a case which we may exclude for the present. To a loop Q_1 going from a around a_1 and back to a and joining two sheets 1–2 of the Riemann surface C_y, there will then correspond a loop Q' going from a to a winding point α and joining the same sheets. Furthermore since a_1 can be brought into coincidence successively with $a_2, \cdots, a_h$, there are loops $Q_2, \cdots, Q_h$ joining these two sheets and going from a around $a_2, \cdots, a_h$. The winding point α may or may not be an ordinary branch point, but in any case, when $y = b$, $\alpha \neq a$.

4. The integral of § 1 may have periods of two natures. The first will correspond to cycles formed by the loops Q_iQ_j, the second to cycles formed by the loops Q_iQ'. Let $\omega_1, \cdots, \omega_h$ be the periods corresponding to the cycles

$$(Q_1 + Q_2, \cdots, Q_h + Q_1)$$

and $\omega_1', \cdots$ the periods corresponding to the cycles

$$Q_1 + Q', \cdots$$

and suppose first that the winding point α comes in coincidence with no other for $y = b$. By the same reasoning as Picard's,[*] it is easy to show then that when y describes a closed path around b, ω_i changes into ω_{i+1} $(\omega_{h+1} = \omega_1)$ and ω_i' into $\omega_i' + \omega_i$. Let $\theta^h = 1$. Then

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[*] Loc. cit., p. 96.

$$\left(\sum_{i=1}^{h} \theta^{ki}\omega_i\right)(y - b)^{-k/h} = R_k$$

is a uniform function of y in the vicinity of b, for all values of the integer k. Taking $k = 1, 2, \cdots, h$ we obtain h, uniform functions of y, and solving the equations for the ω's, we have

$$\omega_i(y) = \sum_{k=1}^{h} \theta^{(i-1)k} S_k(y)(y - b)^{k/h},$$

where the (S)'s are uniform in y, near b. The period ω_i' is of the form*

$$\begin{aligned}\omega_i' = \sum_{k=1}^{h} (y - b)^{k/h}[\varphi_{0k}{}^i + \varphi_{1k}{}^i \log (y - b) \\ + \cdots + \varphi_{ak}{}^i (\log (y-b))^a\} \\ + \textstyle\sum_j (y - b)^{r_j}[\psi_{0j}{}^i + \psi_{1j}{}^i \log (y - b) + \cdots],\end{aligned}$$

this being the general form of the integrals of $E_{2p} = 0$ in the neighborhood of the critical point b. The r's are the roots of the fundamental equation relative to b which are not of the form k/h. When b describes a closed path this integral is increased by one which contains logarithmic terms, or terms in $(y - b)^{r_i}$, as long as all the ψ's, and all the φ's except $\varphi_{01}{}^i, \cdots, \varphi_{0h}{}^i, \varphi_{1h}{}^i$ do not vanish. As we have seen, ω_i' is increased by ω_i, and ω_i contains no logarithmic terms nor terms in $(y - b)^{r_j}$; hence for ω_i' we have the expression

$$\omega_i'(y) = \varphi(y) + \sum_{k=1}^{h-1} \frac{S_k(y)^{\theta(i-1)k}}{\theta^k - 1}(y-b)^{k/h} + \frac{S_h(y)(y-b)}{2\pi i} \log (y-b),$$

φ being uniform in y near b.

5. We may remark that by replacing $P(x, y, z)$ by

$$(y - b)^q P(x, y, z),$$

q being so chosen that

$$\frac{P(x, y, z)(y - b)^q}{F_z'}$$

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* E. Picard, Traité d'Analyse, vol. 3, p. 282. The φ's and ψ's are uniform in the neighborhood of b.

shall be finite in the vicinity of b, we obtain an integral having periods $\bar{\omega} = (y - b)^q \omega$, ω being the period of the integral considered so far, corresponding to $\bar{\omega}$, and that these periods $\bar{\omega}$ are all finite. Hence the functions φ, S uniform near b, have this point for pole, but not for essential singularity, and therefore: *The equation* $E_{2p} = 0$ *has only regular integrals.*

It is seen directly, by considering the circuits on the Riemann surface C_y, that $\sum_{\iota=1}^{k-1} \omega_\iota = 0$. Hence: *In the vicinity of a critical point* b, *there is a certain number of independent periods of the integral*

$$\int \frac{P(x, y, z)dx}{F_z'},$$

say $\omega_1, \cdots, \omega_s$, *having this point either for pole or for critical algebraic point, and such that when* y *describes a closed path around* b, *any period is increased by integral multiples of these.* A period non-uniform near b behaves either like ω_ι or ω_i' or a combination of both. This result really generalizes the one obtained by Picard (loc. cit.). The whole theory of double integrals of the second kind of F can now be developed as in the case of the surface with ordinary singularities. The considerations made there only require that the periods $\omega_1, \cdots, \omega_s$ be finite near b. If this condition is not satisfied for all critical points, we can replace the integral considered by

$$\int \frac{\varphi(y)P(x, y, z)dx}{F_z'},$$

φ being a polynomial in y so chosen that the above condition shall be satisfied for all points b. If ρ_0, ρ, r have the same meaning as with Picard* we have for the number of double integrals of the second kind

$$\rho_0 = N - 4p - (m - 1) + 2r - (\rho - 1),$$

the same formula, if we set $N = \Sigma s$, as that obtained by the French mathematician. N denotes the class when the surface has only ordinary singularities. Since ρ_0 is an absolute invariant with respect to birational transformations, we have by taking this number for the surface $\bar{F}$ birationally transformed of F and with ordinary singularities .

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* Picard-Simart, loc. cit., vol. 2, p. 408.

$$N - 4p - (m - 2) + 2r - \rho = \bar{I} + 2r - \bar{\rho} + 2,$$

where $\bar{\rho}$, $\bar{I}$, $\bar{\rho}$ are the invariants ρ and those of Zeuthen-Segre for $\bar{F}$. If I is the same invariant for F, since $\bar{I} - \rho = I - \rho$, we have

$$I = N - 4p - m = \Sigma s - 4p - m.$$

Hence Σs is equal to the "equivalence" in nodes of the point (a, b, c) in the evaluation of the invariant I for F by means of the pencil C_y. This property can be shown directly for the following cases: 1°. F has only ordinary nodes. 2°. In the vicinity of any of these nodes there lie other nodes, or ordinary infinitesimal multiple curves. In these cases it is easy to show that in the vicinity of the nodes all the numbers such as h are equal to 2. It would be interesting to know if such is always the case, but the preceding investigation shows that for the applications this does not matter.

UNIVERSITY OF KANSAS,
October 17, 1914.

THE FOURTH DIMENSION.

Geometry of Four Dimensions. By HENRY PARKER MANNING, Ph.D. New York, The Macmillan Company, 1914. 8vo. 348 pp.

EVERY professional mathematician must hold himself at all times in readiness to answer certain standard questions of perennial interest to the layman. "What is, or what are quaternions?" "What are least squares?" but especially, "Well, have you discovered the fourth dimension yet?"

This last query is the most difficult of the three, suggesting forcibly the sophists' riddle "Have you ceased beating your mother?" The fact is that there is no common locus standi for questioner and questioned. To the professional mathematician the fourth dimension usually suggests a manifold of objects depending upon four independent parameters, which it is convenient to describe in geometrical language. Occasionally he does not make any use of analysis, but builds up what the Italians call a "Sistema ipotetico deduttivo" of abstract assumptions and conclusions. The whole thing is professional, and unromantic. Such ideas, are, naturally, unintelligible to the "man in the street." Moreover, could he understand them, they would not answer the questions in which he is interested. What he desires is to know whether the fourth dimension, whatever it be, throws any light on the phenomena of spiritism, or the immortality of the soul. Here the professional mathematician is helpless. He is quite as ignorant as is his questioner in regard to the physical existence of a four-dimensional universe enclosing our space of experience. The only sure thing is that neither of them has been there. If the mathematician be wise he will refer his questioner to such books as our present author's "Fourth Dimension Simply Explained"* or E. A. Abbott's classic "Flatland, by A Square" and leave the matter there.

Between the non-mathematicians and the professionals there is a third class composed of amateurs whose interest in the fourth dimension is somewhat different. These are persons of moderate mathematical learning who feel a commendable curiosity as to what geometrical relations obtain in spaces of more dimensions than we know, and are glad to apply such ability as they possess to the study of the question. It is for this class of people, heretofore neglected by English-speaking writers, that the present work was written. Let us say loudly that we believe that the author has succeeded well in meeting their needs. The problems which he takes up are the problems in which such readers are interested, the methods of attack are those with which they are familiar. It must be remembered that the only allowable methods are those of elementary algebra, geometry, and trigonometry, and in the present work we have elementary geometrical methods exclusively. Moreover, these readers know little and care less about systems of logically independent axioms; they will swallow any reasonable number of assumptions with no fear of indigestion; and in the present work very little attention is given to axiom grinding. The book is admirably fitted for its end.

Unfortunately we can not leave the matter there. Whoever writes a mathematical book in these days, especially a book which introduces the reader to new realms of thought, must state clearly somewhere what assumptions he makes, and stand steadfastly by them. He need not, indeed, devote page after page to proving thoroughly familiar theorems by means of some new system of independent axioms, but there should never be any doubt about what facts he has actually assumed. This sort of definite statement is just what our present author never makes; there is always a doubtful zone surrounding his hypotheses. Let us exhibit some examples.

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* New York, Munn and Company, 1910.

Chapter II is introduced (page 74) with these words: "The subjects now to be taken up belong more particularly to metrical geometry; and we shall assume the axioms of metrical geometry, and employ its terms without special definition. In fact, we shall assume all of the theorems of the ordinary geometry, except, for the present, those which depend on the axiom of parallels." (In a note the author refers to Moore and to Veblen for systems of axioms for geometry.) Let us see what conclusions a malevolent reader might draw from this statement. Our author says, on page 23, that two figures intersect if they have a point or points in common. Let us then remember that the leading theorem in solid geometry says that if two planes intersect, their intersection is a straight line. If we hold to this we need read no further, we have excluded a fourth dimension. We must consult the footnote to see how the author would avoid this. As for parallels, how is a reader whose mathematical equipment is such as this work presupposes, to decide whether a theorem really depends upon the parallel axiom or not? The usual proof that all plane angles of a diedral angle are equal is based upon the properties of parallel lines, but an independent proof is easily devised. On the other hand the theorem that all angles inscribed in the same circular arc are equal, does not appear to depend necessarily upon the parallel axiom, yet it does so in fact, and is untrue in the non-euclidean spaces. No writer should leave to his reader the responsibility of settling questions of this sort.

We used the words "non-euclidean geometry," just now; they suggest a second point wherein we take sharp issue with the author. He is very particular to insist again and again that he has made no use of the parallel axiom (before Chapter VI) and that the theorems developed hold equally well in euclidean, and non-euclidean space. Now when an author is careful to exhibit explicitly all of his assumptions, and to refer constantly to them, a statement of this sort inspires confidence; but when he pursues the optimistic course followed in this book, we feel the need of caution in accepting the dictum. Let us be more explicit. On page 27 under the heading *order* we read: "If A and B are two distinct points, then A comes *before* B and B lies *beyond* A in one *direction along the line* AB while B comes before A and A lies beyond B in the *opposite direction*." One naturally concludes that this is an explicit assumption which the author makes about a straight line; but he does not like to put things in such a bald fashion, for he says in the preceding sentence that the relation of order "may be explained somewhat in detail as follows." What he really does is to use *order* to explain *betweenness* and then say, presumably by way of an axiom, page 28: "Given any three points on a line, one of them lies between the other two." Now these assumptions are characteristic of a geometry where a straight line is an open locus, and the reader familiar with non-euclidean geometry would naturally assume that the writer in making such assumptions and building upon them intended consciously to exclude elliptic geometry. Such an inference is the polar opposite of the truth. The author acknowledges in a footnote on page 29 that in elliptic space we have cyclic order on a line, and suggests that the reader either look ahead to page 213 where a new set of axioms of cyclic order are set up, or confine himself to a restricted portion of a line. Let us consider these two suggestions in turn. It is true that on pages 213 ff. the author gives a set of axioms for order in what he calls "double elliptic geometry," i. e., spherical geometry where coplanar lines intersect twice, and if these be carefully followed we can reach the theorems of Chapter I for the spherical case. But he completely overlooks the more usual case of single elliptic geometry, where a straight line is a closed circuit, yet coplanar lines meet but once. Here we encounter a really serious difficulty. We naturally say that a point is *between* two others if it lie upon the smaller segment of their common line, assuming the two segments not equal. Let us then take two coplanar lines and mark on one of them three points ABC in such fashion as to divide the whole length into three equal parts. The other line will then contain a point of just one of the three segments BC, CA, AB. Now move one of the three points a very slight distance from the line, without removing it from the plane. We have a highly attenuated triangle ABC, and a line coplanar therewith which intersects one side, and two sides produced. But this is in flat contradiction to the axiom of Pasch, "A line intersecting one side of a triangle, and another side produced, intersects the third side," upon which the whole book rests.

The author would probably reply that the axiom of Pasch was not necessarily true in the whole of the elliptic plane, but only in a *restricted region*. Let us see what is meant by this phrase which is cropping up continually; as on page 153, where a certain theorem is announced for the plane but a footnote says: "Or at least *in a restricted portion of a plane*." Just what does our author mean by this oft recurring phrase? We turn to the index which refers us to page 19 and here we are referred to page 6 of the author's "Non-Euclidean Geometry."* There we read: "The following propositions are true, at least for figures whose lines do not exceed a certain length. That is, if there is an exception, it is in a case where we can not apply the theorem, or some step in the proof, on account of the lengths of some of the lines. For convenience, we shall use the word *restricted* in this sense, and say that the theorem is true for restricted figures, or in a restricted portion of the plane." Surely, these are dark sayings. The meaning seems to be that certain theorems are usually true, and if they are not true, why that is because some of the lines involved are too long. But how are we going to tell in any particular case whether this difficulty is going to arise or no? We ask in vain; the author vouchsafes no reply. What he has in mind is, probably, something of this sort. Let us start with a complete elliptic space, and then restrict ourselves to such a region as the interior of a sphere whose radius is not greater than one quarter of the total length of a line. In this region two points determine a single definite segment, every segment may be extended beyond either end, points on a straight line have an open order, and Pasch's axiom is true. We seem to have removed all of our difficulties at a stroke. Alas no! In this vale of tears nothing is obtained without cost, and the price which we have paid has been to sacrifice the right to extend a given segment by a preassigned amount. Equally disagreeable is the fact that we can no longer surely drop a perpendicular on a line from an outside point. We are, in fact, in a very parlous state.

No, we take issue entirely with the author on the whole non-euclidean question. We feel that the book would have been stronger and better if he had taken his stand frankly at the outset on some recognized system of euclidean axioms. Few readers of the type for whom the book was written would have missed the non-euclidean cases, fewer we believe than the number of those who are in danger of being confused by the present arrangement. Such a course would have entailed some modification in the highly ingenious treatment of the regular hypersolids, but this handicap would have been far outweighed by the gain in clearness and precision.

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* Boston, Ginn and Company, 1901.

The work begins with an introductory chapter giving a historical account of the subject. The bibliographical references are many, and seem to have been compiled with care. The apology for the exclusive use of elementary synthetic methods is less convincing. The real reason for such a course is that the class of readers especially interested in the subject matter understand no other methods. The author is just right in saying (pages 13 ff.) that the study of the higher spaces throws a flood of light upon the lower ones, but such a statement is far more convincing if illustrated by a few concrete analytical examples. For instance, the statement that the system of spheres in our space gives a sensuous representation of the points in four dimensions does not drive the matter home half so convincingly as when we point out that if we refer the point (X, Y, Z, T) in euclidean S_4 to the oriented sphere in euclidean S_3 whose center is (X, Y, Z) and radius $-\sqrt{-1}T$, the distance of two points in S_4 will have the same analytic expression as the length of a common tangent of the corresponding spheres.

In Chapter I we have the systematic foundation for four-dimensional geometry. The axiom of Pasch, already mentioned, is fundamental in all of this work. The plane, hyperplane (S_3), and four-dimensional space are actually built up by a series of triangle transversal constructions. This is what has come to be recognized as the standard method, in recent years, and the present author employs it with skill and success. A good deal of attention is given to convex figures, which are carefully defined, though it is not clear whether the author is familiar with the recent work of Whittemore and Lennes dealing with such. The chapter ends with a discussion of various graphical properties of solids and hypersolids. Some of the latter are more easy to understand than others. We pass naturally from the pyramid in S_3 to the hyperpyramid in S_4. Much more elusive are the double pyramids and double cones (pages 66 and 71) obtained by drawing lines from the points of the interior of a closed plane curve, to those between two points not in the same S_3 as the curve.

The author's avoidance of existence postulates gives a startling sound to some of his statements, as this on page 59: "*A space of four dimensions* consists in the points we get if we take five points, not points of one hyperplane, etc." The reader almost instinctively says: "Dear me, this is so sudden!" Some definitions are also open to objection as, page 32: "A point is said to be *collinear with a triangle* when it is collinear with any two points of the triangle." Leaving aside the secondary consideration that it is a little inelegant to use the word *collinear* in connection with a two-dimensional figure, it may be contended with some force that according to this definition a point is collinear with a triangle when it is collinear with *each* two points thereof. We reach the author's true meaning if we omit the word *any*.

In spite of these criticisms we feel that the reader who has read the present chapter understandingly, knows a good deal more about geometry than he did before.

The second chapter is devoted almost exclusively to perpendicularity and is decidedly interesting. The author makes very clear the distinction between *simply* perpendicular planes, which lie in an S_3 and are the usual perpendicular planes of commerce, and *absolutely* perpendicular planes where each contains a pencil of concurrent lines perpendicular to the other. It is perhaps a pity that he refers only in a note (page 85) to *half-perpendicular* planes, each of which contains just one perpendicular to the other. It is regrettable that one serious misstatement recurs several times in the chapter. For instance, we read on page 77, theorem 3: "*Through any point outside a hyperplane passes one, and only one, perpendicular to the hyperplane.*" How can a writer so desirous to "bless and preserve to our use" elliptic space appear to forget that there all lines through a point might be perpendicular to a hyperplane? An equivalent mistake will be found on pages 81 and 82. Another slip occurs in the theorem on page 94. If two planes are not in an S_3 the lines in one coplanar with lines in the other have been defined as *linear elements*. We read: "*Given two planes, not in a hyperplane, if any two of their linear elements have a common perpendicular line, they* *all have a common perpendicular hyperplane.*" Leaving out the question of the ambiguity of the word *any,* the author clearly means a common perpendicular line in their plane, for two intersecting lines have plenty of common perpendiculars not in their plane.

Chapter III is devoted to various sorts of angles. It starts with a proof that two skew lines have always one common perpendicular. This is highly ingenious, being based on considerations of continuity and irrational numbers; of course the author has cut himself off from the usual simple proof based upon parallels. We next come to a study of the angles of two planes which have but one common point, and a discussion of the curious figure of isocline planes which, though only intersecting once, have yet an infinite number of equal minimal angles. They are dual figures to the Clifford parallel lines of elliptic space. The author pays some attention to what he rather infelicitously calls *point geometry,* i. e., the geometry of a bundle of concurrent lines, and *edge geometry,* generated by coaxal planes. The two geometries are dual to one another, and have elliptic measurement.

The fourth chapter takes up the concepts of symmetry, order, and motion. This, to the reviewer, is by far the least satisfactory chapter of the book. To begin with, the language is frequently so obscure that the reader vainly tries to find out what is assumed, what is defined, and what is proved. For instance we read on page 160: "We shall say then that the order of a triangle can not be changed by any motion of the triangle in its plane, regarding this statement as, in part, a definition of the phrase *motion in a plane.*" The reviewer's guess is that this means, motion in a plane is a transformation of the plane which leaves invariant the distance of each pair of points, and the order of each triangle. What, then, is the order of a triangle? We read (page 154): "We have two principles on which we can base the theory of order in a plane:

"*I. A and B being any two points of a plane, a point which is on one side of the line AB is on the opposite side of the line BA.*

"*II. O, A, and B, being any three non-collinear points of a plane, B is on one side of the line OA and A is on the other side of the line OB.*"

Exactly what rôle in the drama is played by a *principle* the reviewer does not know. The natural inference is that we have here a definition of the phrases *same side, and opposite* *sides.* Yet this can hardly be the case for we can not say of a definition that it is true or not true, yet we continually meet statements like that in the corollary (page 155): *If I holds true of BC and II holds true of OA and OB, then II will hold true of OA and OC.* The reviewer has had lucid intervals when he believed that he understood what these words meant, but they were infrequent and of uncertain duration. Similar statements occur frequently in the following pages. A very definite error occurs in the theorem of page 170: "*If after a motion of a hyperplane on itself there is no point which occupies the position that it occupied before, then every point occupies a position that could have been reached by the motion on itself of some plane of the hyperplane, or by a screw motion.*"

The proof begins as follows: "Let A be the first position, and B the second position of some point, let C be the second position of the point whose first position was B, and let D be the second position of the point whose first positions was C. We will assume that A, B, C, and D are not collinear." "But," asks the careful reader, "what right have you to assume that these points are not collinear?" The answer is "None whatever." As a matter of fact, in the most striking form of motion of that elliptic space which is so dear to our author, the points are collinear, for each (real) point moves along that line of a congruence of Clifford parallels which passes through it, and each plane rotates about one such line. This error invalidates the proofs of the two important theorems on page 174.

Chapter V is devoted to hyperpyramids, hypercones, and hyperspheres, and calls for no special comment. In Chapter VI we have, for the first time, the explicit euclidean assumption about parallels, and an adequate discussion of parallel planes, half-parallel planes, and parallel hyperplanes. The treatment is good, but would have been better if the author had omitted the section on the "hyperplane at infinity." He begins (page 230): "We express certain facts of parallelism as if they were matters of intersection, from which, indeed, they are derived by limiting processes. Thus we say that two lines intersect at infinity only as another way of saying that they are parallel." This is just exactly right. Now we know what we mean by "intersect at infinity," we have as yet, however, no meaning for the phrase "point at infinity." If two lines intersect in the usual sense, they have a common point, if they intersect at infinity they have no common point, so that "point at infinity" must mean some other undefined thing which they share. Yet our author goes serenely on: "Points at infinity are sometimes called *ideal points*. A line has a single *point at infinity*, its intersection with any parallel line." The strange thing is that it is perfectly easy to put the whole subject upon a satisfactory basis. We begin by defining a point at infinity as a bundle of parallel lines, and say that an infinite point lies on a (finite) line, if the latter be a member of the bundle defining the former. Starting thus, the geometry of the infinite domain can be built up in a simple and rigorous way. The latter part of the chapter goes to hyperprisms and double prisms, hypercylinders, and double cylinders. The treatment is careful, but the subjects are sometimes difficult to grasp, especially the double prisms and double cylinders.

Chapter VII is the important metrical chapter, and is entirely devoted to volumes and hypervolumes. The author determines carefully, and with not a little ingenuity the limiting volumes, and the hypervolumes of all of the more important hypersolids. It is regrettable, though perhaps not surprising, that, as usual, he shies at definitions. For instance, we read (page 270): "A hypersolid is supposed to have a *hypervolume* which can be computed from the measurements of certain segments and angles, and which can be expressed in terms of the hypervolume of a given hypercube, taken as a unit." Here again, a few simple assumptions would put the matter in a much better light. Hypervolume is a numerical coefficient attached to a hypersolid. Congruent hypersolids have the same hypervolume, and if a hypersolid be the sum of two others, its hypervolume is the sum of theirs. It would probably be well also to assume the DeZolt theorem, a hypersolid can not have the same hypervolume as a part of itself, and perhaps also the proposition that an infinitely short hyperprism has an infinitesimal hypervolume.

Chapter VIII is the last and deals with the regular hypersolids. Four of these, the pentahedroid, hypercube, 16-hedroid, and 24-hedroid are immediately reached, and easily discussed. Then follows a half-hearted discussion of the Eulerian formula for hypersolids

$$\text{No. of vertices} + \text{No. of faces} = \text{No. of edges} + \text{No. of cells.}$$

We say that the discussion is half-hearted because the author says in a note on page 302 that the formula will not be used again, and may be omitted. The formula is said only to hold for *simple* polyhedrons, though we are never informed exactly which these are. We are, indeed, told on page 63 that the term "polyhedroid" will be applied only to certain simple figures defined individually. This promise is never carried out, and we are led to the idea that the formula is true for simple polyhedroids, since simple polyhedroids are those for which it is true. It would have been better to give more precise definitions or else leave it out entirely. For instance, the following hypersolid is simple, in the sense that it is easily described. We start with a rectangular hyperparallelepiped (heaven save the mark!) whose dimensions are $3 \times 3 \times 3 \times 1$. This can be constructed from 27 abutting unit hypercubes, and fulfills the formula above. We now construct a "hyperdoughnut" by removing the middle hypercube. By this process we add to our original hypersolid all of the vertices, edges, and faces of the hypercube, and all but two of its cells. The formula is no longer applicable.

The author next passes to nets of regular polyhedrons, meaning thereby systems which entirely fill our space. His wording is here sometimes unfortunate, so that the reader is in danger of being confused. Let us cite two instances. The first is page 305: "The only sets of regular polyhedrons that can be used to form nets are 4 tetrahedrons, cubes or dodekahedrons at a point," etc.

The reader naturally concludes that space could be filled with cubes abutting in fours, and wonders how these would appear. But what the author means is that if space *can* be filled with regular polyhedrons meeting four at a point, these solids must be tetrahedrons, cubes, or dodekahedrons. An even more cryptic utterance occurs on page 306: "Any combination which more than fills the part of euclidean space about a point belongs to hyperbolic geometry, and any combination which does not fill the part of euclidean space about a point, belongs to elliptic geometry." These mystic words cover a highly ingenious bit of geometrical reasoning, which is worth explaining.

We start with the geometry of a bundle of concurrent lines in S_3, defining the angle of two such lines, after Laguerre, as a constant multiple of the natural logarithm of a cross ratio which they determine with two generators of a certain cone. This definition is the same in euclidean and the classical non-euclidean geometries, hence the geometry of lines of a bundle is the same in all. We next take a regular tetrahedron, and drop half-lines from its center perpendicular to its faces. These will determine four trihedral angles which together fill up the space about this center. We then wonder whether these trihedral angles are congruent to those of a regular tetrahedron, so that the space could have been equally well filled by four abutting tetrahedrons, but we find that in euclidean space the face angles of these central trihedral angles are greater than those which appear in a regular tetrahedron. Not so in elliptic space. Here the area of a triangle is measured by the excess of the sum of its angles over π. If we take our regular tetrahedron large enough, the face angles will be just those of the four central trihedral angles. We thus complete the first step in the proof that elliptic space can be completely filled by regular tetrahedrons, meeting in fours. The author closes with an account of two pleasant little figures, the 600-hedroid, and the 120-hedroid. The treatment is admirable considering the complexity of the subject.

What shall we offer as our final opinion? We have praised the book in general and damned it enthusiastically in detail. Let the praise be remembered and the blame forgotten. Our author has written a book well fitted to interest and stimulate the audience he had in mind. He has done far more for his day and generation than has Dryasdust, who never makes a mistake because he never has anything to say.

J. L. Coolidge.

SHORTER NOTICES.

Die neuzeitliche Entwicklung des mathematischen Unterrichts an den höheren Mädchenschulen Deutschlands inbesondere Norddeutschlands. Von Prof. Dr. J. SCHRÖDER. (Abhandlungen über den mathematischen Unterricht in Deutschland veranlasst durch die internationale mathematische Unterrichtskommission, herausgegeben von F. KLEIN. Band 1, Heft 5.) Leipzig, Teubner, 1913. xii + 183 pp.

THE publications of the international commission on the teaching of mathematics are especially significant at the present time when some persons are challenging the right of mathematics and other so-called disciplinary studies to continue to hold their time-honored positions in our curricula. As Germany is one of the strongholds of experimental psychology and as many of the experiments relating to "transference" of power from one subject to another have been made by Germans, it is especially interesting to see what effect all these discussions have had upon the amount and kind of mathematics in the courses of study of the schools of that country. The German subcommittee have planned five large volumes of "Abhandlungen." Two of these deal with their higher schools; the other three treat of separate questions in the instruction in higher mathematics, of mathematics in the technical schools, and of elementary and normal school mathematics. There will be almost forty separate pamphlets, written by men who are authorities in their several lines. The editor-in-chief is Professor Klein, who is undoubtedly better fitted than any one else to fill this responsible position.

The book under review is the fifth and last part of the first volume of the series. Of the three sections into which it is divided, the first two deal with the development and present conditions in "northern" Germany and the third takes up the present status in other parts of the country. The first section gives, in a concise form, a picture of the very primitive conditions in women's education that prevailed until recently, and it traces out the principal influences which have contributed to an amelioration of these conditions. The reader's attention is held by the stirring accounts of the long and hard struggle against ignorance and prejudice waged at first by a few far-seeing and brave pioneers and later by bands of men and women associated together for the purpose of establishing adequate schools for girls. The ordinance of May 31, 1894, was a great victory in many ways, but it left much to be desired. This was particularly true from a mathematical point of view, for substantially all mathematical instruction except practical arithmetical calculations were explicitly excluded from the curriculum. The defects in these regulations seemed to stimulate the friends of the education of girls to a renewal of their efforts.

Fourteen years later the Prussian government issued new regulations putting the girls' schools on practically the same basis as the boys' schools as regards the governing boards and the training and rank of the teaching staff. The length of the course was increased and the curriculum was strengthened. The work in mathematics was, briefly, as follows: In the Lyzeum (which the girl entered at the age of six for a ten years' course) the requirements were (1) a very thorough drill in arithmetic, especially in mental arithmetic and in the arithmetic calculations of every day life; (2) algebra to quadratic equations with one unknown; (3) plane geometry to the theory of the circle; (4) calculations of the surfaces and volumes of the simple solids. Those who wished to prepare for elementary teaching entered the Oberlyzeum, where the mathematics for the first three years consisted of (1) arithmetic and algebra to the theory of complex numbers and equations of the second degree in two unknowns (with the binomial theorem for positive integral exponents); (2) plane geometry to the theory of harmonic points and lines; (3) plane trigonometry; (4) stereometry with regard to the principles of projective drawing; (5) the elements of plane analytic geometry. The last year was given to "practical work." Those who took the two years' course in the Frauenschule discontinued the study of mathematics after leaving the Lyzeum. Those who wished to prepare for university study could enter any one of the three kinds of Studienanstalten, leaving the Lyzeum at the end of the eighth or seventh year according to whether they went to an Oberrealschule or a Realgymnasium or a Gymnasium. In each of these institutions the course in mathematics was more extensive than that offered in the first three years of the Oberlyzeum. The maximum amount was offered in the Oberrealschule, where, in addition to the work given in the thirteen years of the Lyzeums, there were (1) equations of the third degree, (2) the most important infinite series, (3) analytic and synthetic treatment of conics, (4) sufficient spherical trigonometry for an understanding of mathematical geography.

As compared with the order of 1894, this was a wonderful gain for mathematics. But even yet this subject is not up to the standard for the corresponding schools for boys. An increase in the number of hours given to mathematics in the Lyzeum and the introduction of at least the elements of the calculus into the Studienanstalten are amongst the changes strongly urged. But in view of the rapidity of development in the last two decades, it is probably only a question of a few years until the work in mathematics in the girls' schools is equivalent to that in the corresponding institutions for boys.

In the third section of the book, the Prussian conditions are compared with those in the various other German states. For one who is not a German the questions taken up here are not so vital as those which preceded.

It must be noted that the discussions in this book are confined to the mathematics in schools below university grade. The "Study of Mathematics in the German Universities since 1870" is the subject of a separate pamphlet in the third volume of this series. The author's genuine sympathy with the movement for equality of educational opportunities for girls and boys is evident on almost every page. All persons interested in the education of girls, whether from a mathematical point of view or not, are indebted to Professor Schröder for this valuable book, with its extensive list of references to monographs, journals, and government publications.

Of the present attitude of the governing bodies towards mathematics in schools for girls, he says in his conclusion: "All the German states are convinced of the necessity of proper mathematical instruction in the higher schools for girls. In all the recent regulations there are definite directions that, through participation in mathematical instruction, the girls shall be taught to think more clearly and to express their thoughts in a simple and direct manner."

E. B. COWLEY.

Calcul numérique. Par R. DE MONTESSUS et R. D'ADHÉMAR. Paris, O. Doin et Fils, 1911. 246 pp.

THIS work is in two parts.

The first part (pages 3–138) is by R. de Montessus and is entitled Opérations arithmétiques et algébriques. Its principal content is in the three chapters devoted to the calculation of the roots of numerical algebraic equations and of transcendental equations. For the resolution of these problems a rich variety of methods is given. Many of these are elementary and are usually found in texts on the theory of equations; but others of them are less generally employed. As examples of the latter are those (in Chapter V) in which the calculus of differences is employed for numerical approximations. The methods given are illustrated with many numerical exercises. This part contains a very convenient and readable account of its subject matter.

The second part (pages 141–237) is by R. d'Adhémar and is entitled Intégration. It is concerned primarily with quadratures and differential equations, but contains a short digression on implicit functions and equations, the latter being treated by the method of successive approximations. The author has attempted only an introduction to his subject; this is desirable and is indeed all that could be done in the short space employed.

R. D. CARMICHAEL.

Elementary Theory of Equations. By L. E. DICKSON. New York, John Wiley and Sons, 1914. v + 184 pp.

IT is a good omen for the development of mathematics in this country when some of our most brilliant men of research are willing to take the time to prepare elementary texts suited to the needs of beginners. It is therefore a pleasure to welcome this excellent book, by Professor Dickson, on the elementary theory of equations.

Naturally, no treatment of the difficult Galois theory is given. On the subject of invariants one finds merely a few illustrative examples and no systematic exposition. These omissions will probably be approved by every one who desires to use the book in his classes. The remaining and more elementary aspects of the theory of equations are developed with sufficient fulness to meet the needs of all teachers who are likely to employ such a book for purposes of instruction.

Certain sections, with a total of about fifty pages, are marked with a dagger; the remaining sections form an independent whole and provide a briefer treatment for those who desire such a course.

A large number of illustrative problems are solved in the text, and about five hundred others are given as exercises. These are carefully selected and graded and are distributed in such a way as to be convenient for purposes of instruction. They are so placed that a reasonably brief and elegant solution may usually be expected.

Interspersed throughout the text is a considerable number of brief and clear statements which will serve to give the student an outlook upon various topics of decided intrinsic and historical interest.

In Chapter I (pages 1–17) is given a treatment of the scientific art of graphing. By its aid one may make better graphs in less time than otherwise and at the same time draw negative conclusions so as to sense more than he sees.

Chapter II (pages 18–30) contains a satisfying introduction to complex numbers together with an excellent and concrete discussion of the roots of unity. Chapters III (pages 31–37) and IV (pages 38–46) are given to a treatment of the solution of cubic and quartic equations respectively.

An exposition of the first proof by Gauss of the fundamental theorem of algebra is contained in Chapter V (pages 47–54), while Chapters VI (pages 55–62) and VII (pages 63–80) are given respectively to elementary theorems on and symmetric functions of the roots of an equation.

Chapter VIII (pages 81–92) contains a treatment of reciprocal equations, construction of regular polygons, duplication of cubes, and trisection of angles. It is shown that the second last and in general the last constructions are not possible with ruler and compass. Likewise the impossibility of such a construction for a regular polygon of 7 or of 9 sides is proved by a method applicable to the general case of nonconstructible regular polygons.

Chapter IX (pages 93–108) is devoted to the problem of isolating the real roots of an equation with real coefficients. In view of the errors which are often found in the statements and proofs of the theorems of Descartes and Sturm and Budan, it is refreshing to find them here treated in a satisfying manner.

In Chapter X (pages 109–126) on the solution of numerical equations emphasis is put upon the method of Newton on account of its several advantages; but an appropriate treatment is also given of other methods.

Chapter XI (pages 127–149), which is independent of the earlier chapters, contains an easy introduction to determinants and their application to the solution of systems of linear equations.

Finally, Chapter XII (pages 150–166) is devoted to the theory of resultants and discriminants.

The reviewer believes that this book will be found highly satisfactory and that it will have wide use.

R. D. CARMICHAEL.

The Teaching of Geometry. By DAVID EUGENE SMITH. Ginn and Company, 1911. v + 339 pp.

THE Teaching of Geometry is a good book. It is "worth the paper it is printed on," which can not be said of some books and it is worth a great deal more, which can be said of comparatively few books. The preface puts the reader in the spirit of the text immediately. The table of contents tells as completely as one page can what is contained in the three hundred and thirty. The first chapter contains a discussion of "questions at issue" in mathematical pedagogy. The reader is given both opportunity and encouragement to range himself on one side or other of these questions, but not without an understanding of what they mean. The next five chapters include some reasons for studying geometry, a brief history of the development of the subject giving necessarily much prominence to Euclid's part in this development and to efforts at improving or modifying his treatment of the subject. The remainder of the book takes up some of the details of classroom work, but no cut and dried rules are given. The author's own words describe the spirit of this text quite excellently: "Get a subject that is worth teaching and then make every minute of it interesting." J. V. MCKELVEY.

A History of Japanese Mathematics. By DAVID EUGENE SMITH and YOSHIO MIKAMI. Chicago, The Open Court Publishing Company, 1914. v + 288 pp.

THE history of Japanese mathematics as given by Smith and Mikami seems to furnish a parallel in some respects to the history of mathematics in Europe. Problems of a similar sort were studied in the east and west by very different methods with uniformly identical results. The Japanese methods seemed to be painstaking and exquisite in detail, rather than powerful and rigorous as were the European. The "circle principle" similar to the calculus, an equivalent of Horner's method, the value of π, determinants, configurations of circles, ellipses and straight lines on a folding fan, i. e., in the sector of an annulus were among the subjects studied by Japanese mathematicians. In the earlier years, foreign learning was forbidden entrance to Japan. A few pupils studied with teachers of their own choosing. Frequently these teachers kept secret their most precious discoveries or revealed them to their favorite pupils on condition that the information should go no further. Japanese teachers were accustomed for a long time to post problems for solution on the temple doors, frequently using a pupil's name rather than their own.

The "closed door" method of study continued, but with weakening prestige, until the beginning of the nineteenth century, when western learning was freely recognized.

J. V. McKELVEY.

Die realistische Weltansicht und die Lehre vom Raume. Von E. STUDY. (Band 54, Die Wissenschaft, Einzeldarstellungen aus der Naturwissenschaft und Technik). Braunschweig, Friedr. Vieweg und Sohn, 1914. ix + 145 pp.

THE purpose of this essay appears very clearly from an announcement by the publishers, according to which the author attempts to show that the question concerning the true nature of space, like others, is a problem of natural science that cannot be answered by "pure reason," as proposed by Kant.

In the theory of the problem of space it is shown that, as in natural sciences in general, nothing can be accomplished without the aid of experience and hypotheses. Of these, so far, only the euclidean hypothesis has gained practical importance, although to the non-euclidean hypotheses must be accorded the same theoretical value (Erkenntniswert).

The first part of the book deals with philosophic questions and is accessible to a generally educated public. The second part requires a fair knowledge of analytic geometry. It is the contention of the author that a sufficient and precise formulation and understanding of the theory of space is impossible without analytic geometry.

Throughout the book Study lives up to his well-merited reputation as an acute and competent critic.

In the first chapter the problem of space is presented from the standpoint of realistic philosophy to which the author confesses. It is headed by a motto due to Helmholtz: "Unwürdig eines wissenschaftlich sein wollenden Denkers ist es, wenn er den hypothetischen Ursprung seiner Sätze vergisst." According to this philosophy, space is real and not merely a concept of logic. We conceive space because we live in it, and we are not able to say exactly what it is. To conceive it in a mathematical sense we must make certain hypotheses which are based upon experience.

The second chapter deals with the enemies of realism: the idealists, the positivists and, last but not least, the pragmatists; it may be called a clever satire against these philosophies. Pragmatism in particular is torn to shreds and, according to Study, does not seem to reflect great credit upon the intellectuality of the age and class that conceived it. He speaks of the "gallertartige Konsistenz dieser Quallenphilosophie" (jelly-fish philosophy).

After critizing the idealistic (a priori) conception of space, in the succeeding fourth and fifth chapters the author develops in detail the realistic theory of space. From the data of experience alone (positivism) it is not possible to deduce a system of mathematical concepts, or a system of geometry. What we here may expect from experience and the construction of hypotheses, in particular of the experiment, is not more nor less than in other problems of natural science. In many cases, when making hypotheses, we may not expect a uniform answer from nature. The hypotheses on the nature of empiric space are of exactly the same type as all other hypotheses of the natural sciences. From this standpoint the only admissible hypotheses of natural geometry that may be considered seriously are the euclidean and the so-called non-euclidean geometries. The reason for this is that for these we have sufficient material for induction, while this is not true of certain "artificial" geometries. The different steps in the mathematical treatment of the problem of space from the realistic standpoint are carefully and clearly explained.

Naturally, such a treatment leads to a sharp conflict with the school of "logisticians" and enthusiasts of "axiomatics." This phase of the great problem is considered in the concluding chapter: axiomatics in geometry. It seems to me of great importance and actuality and I should like to advise every young American mathematician to read carefully this clear cut presentation of the issue.

On the tendency to reduce all and everything in mathematics to axiomatics the author has this to say:

"Mit Uebertreibungen dieser Art, die eine etwa als Axiomasis zu bezeichnende wissenschaftliche Modekrankheit darstellen, haben wir es jedoch nicht zu tun. Man muss sie austoben lassen: Gleich allen Moden werden sie von selbst aufhören."

Concerning the comparative value of productive mathematical activities Study makes the following statement:

"Auf die Resultate kommt es vor Allem an, und in zweiter Linie erst steht die Methode für Den, der nicht nur mathematische Philosophie oder philosophische Mathematik treiben, sondern sich schöpferisch betätigen will."

The critical remarks concerning Poincaré's scientific activity, so far as they are of a personal nature, might have been omitted. Poincaré was precisely of the type of mathematicians that were after results; with him the method was of secondary importance.

As a whole, the reading of the book with its vigorous and aggressive style is very refreshing, and nobody that intends to be well informed on the foundations of mathematics should fail to familiarize himself with its contents.

ARNOLD EMCH.

Contribution à l'Étude des Courbes convexes fermées et de certaines Courbes qui s'y rattachent. Par CHARLES JORDAN et RAYMOND FIEDLER. Paris, A. Hermann et Fils, 1912. 77 pp.

As the authors point out, this monograph is an auxiliary of their investigations in geometric probabilities and treats of closed convex curves and some associates that may be conveniently established by the method of tangential polar coordinates.

Such a system in a plane may be defined as follows:

Choose a point as the pole and a straight line through it, with a definite direction as the positive direction, as the polar axis. Then, a directed straight line will be defined:

(I) When the angular coefficient α is given; i. e., the angle through which the polar axis must be turned in a counter clock-wise direction to make it coincide with the positive direction of the given straight line. Thus, α may vary from 0 to 2π.

(II) When the distance $| p |$ of the pole from the straight line is given; the tangential vector p will be considered as positive or negative according as the pole lies to the left or right of the straight line; p may therefore vary from $-\infty$ to $+\infty$.

An equation $F(p, \alpha) = 0$ under certain conditions represents a curve enveloped by the straight lines whose coordinates α and p satisfy this equation. Also in this system of coordinates the parametric representation (uniformization) is of great importance. In order that $p = f(t)$, $\alpha = \varphi(t)$ represent a curve of type II (including convex curves) it is necessary that $f(t)$ and $\{\varphi(t) - \pi t/w\}$ be coperiodic uniform functions of t, of period $2w$, and that the first derivatives of p and α exist for all values of t. Moreover, α must be a monoton function of t.

By means of these coordinates and parametric representations, such curves as reciprocal polars, "developoids," "developants," "tangential radials" and "antiradials," parallel curves, envelopes of diameters, "centrics" and "medials" of convex curves, and "orbiforms" (convex curves of constant diameter) and their relations are investigated.

For the purpose in view the method of tangential polar coordinates is very effective and, in the hands of the authors, has produced a number of new results.

The treatment is clear and concise, but the lack of appropriate headings throughout the text detracts somewhat from a convenient and systematic presentation of the subject.

Typographically the work is below the usual French standard of book-making. See, for instance, the formula below the second line on page 25.

ARNOLD EMCH.

Natural Sines to Every Second of Arc, and Eight Places of Decimals. By EMMA GIFFORD. Published by Mrs. Gifford, Oaklands, Chard, Somerset, 1914. vi + 543 pp. Price £ 1.

UNTIL the opening years of the twentieth century it seemed quite unnecessary to undertake the computation of tables of natural functions beyond what had long been in print. The recent development of various types of calculating machines, however, has destroyed the monopoly held by logarithms for the past three hundred years, and has restored to the natural function something of its early prominence, particularly in the work in astronomy. When Georg Joachim, surnamed Rheticus (b. 1514), computed his great table to every 10″ of arc and to ten figures, it was thought that nothing further could well be demanded, and the posthumous publication of these tables by Valentine Otho, under the title of Opus Palatinum (1596) was justly felt to mark a great epoch in mathematical progress. The invention of logarithms only a few years later, however, relegated the work of Rheticus to a position of relative insignificance, and there it would have remained had not the rapid progress of calculating machinery in recent years rescued it from this unhappy position.

In 1897 Jordan published his table of natural sines, which was a reprint of Rheticus to every 10″ of arc and to seven figures. This was the first important evidence of the return of the natural function to its former position, but it merely made accessible an important part of the Opus Palatinum, so that it represented nothing new in its line.

Mrs. Gifford started out to work de novo on a table of natural sines to every second of arc, not having access to the work of Rheticus when she began. By the expenditure of an amount of time and work which seemed out of all proportion to the results secured she computed two hundred and forty sines. She then secured a copy of the Opus Palatinum and proceeded to find the sines by interpolation, checking from the results she had already secured and from Callet's centesimal table in which are given a thousand sines to the quadrant, or one to every 324″ of arc. When in doubt as to the eighth decimal place she checked by the equation $\sin^2 x + \cos^2 x = 1$. The work of interpolation was performed by the aid of a Thomas calculating machine, using the figures given by

Rheticus for every 10″ of arc, and carrying these to ten decimal places as in the original table.

The arrangement of the table is not semiquadrantal, as in those in common use. This latter form does not lend itself easily to a large octavo page when the work is carried out to every second. Otherwise the arrangement is similar to that found in Chambers and other familiar tables. A convenient table of differences is given in the margin for each half of each page,—that is, for every 5′ of arc.

As to the accuracy of the work it is too early to speak. A set of tables containing a million and a half figures is sure to have errors, particularly as nine of the ten columns on a page represent new calculations. On the other hand Mrs. Gifford is a careful and experienced computer, she has had the aid of the best machinery and tables, and she has checked her work with care, so that it is probable that the number of misprints and errors in calculation has been reduced much below that found in the older type of tables.

Aside from the recent work of M. Andoyer, no such notable contribution to this kind of mathematical literature has been made for many years, and Mrs. Gifford is to be congratulated upon the completion of her labors in this important field. The tables should be in the library of every higher institution of learning, and in every astronomical and mathematical laboratory. DAVID EUGENE SMITH.

Applied Mathematics. By H. E. COBB. Ginn and Company, 1911. vii + 274 pp.

WE meet many calls for "real problems" and "useful mathematics" but too often there is a failure to distinguish carefully between what is useful, what is real, and what is concrete. Carson* has ably drawn the distinction between these terms and forcefully argues that real mathematics is what we need. "The essence of reality is found in definite recognizable percepts or concepts, and is therefore a function of the individual and the time"; that is, reality depends upon the past experiences of the individual and not only upon the subject matter. It is doubtful whether the author of the book under review has kept this important distinction in mind, but the teacher may select such parts as represent reality for his particular students.

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* Essays on Mathematical Education, Ginn and Co., p. 35.

The book is a collection of problems intended to be useful in the student's future work. The use of such problems the author believes will also correlate the several mathematical subjects among themselves as well as with physics.

The problems are grouped under the following heads: measurement and approximate number, vernier and micrometer calipers, work and power, lever and beams, specific gravity, geometrical constructions with algebraic applications, the use of squared paper, functionality, maximum and minimum values, algebraic solution of geometry problems, logarithms, the slide rule, angle functions, variation, exercises in solid geometry, heat, electricity, and logarithmic paper. In the appendix are tables of unit equivalents, four-place logarithms, and a bibliography of problem sources.

Some problems require the student to obtain his own data by measuring, weighing, etc. Many problems contain two sets of numbers, one set leading to integral results and one involving fractions. The part on numerical calculation and approximation deserves special mention. Necessary definitions and a minimum of theory precede each set of problems. This collection is just such a set of problems as live teachers wish to have at hand for frequent selection of those which may be *real* for their students.

ERNEST B. LYTLE.

Handbuch der angewandten Mathematik. Herausgegeben von H. E. TIMERDING. Berlin and Leipzig, B. G. Teubner.

Vol. 1. *Praktische Analysis.* Von H. v. SANDEN. 1914. 185 pages.

Vol. 2. *Darstellende Geometrie.* Von J. HJELMSLEV. 1914. 320 pages.

ONE or two decades ago there was considerable agitation to have mathematics taught to technical students by engineers rather than by professional mathematicians. The reaction against this tendency, in which the Perry movement played an important part, has been decisive and possibly extreme. Now every reputable technical school in Europe and America has its mathematics taught according to mathematical standards by men trained as mathematicians.

But the engineers had a legitimate grievance; the theoretically trained man was too frequently unable to apply his knowledge to concrete problems. The reason the Perry movement failed was that the theoretical side was largely replaced by empirical procedures, thus making it impossible for the student to take the initiative. All he could do was to follow blindly the rule of thumb.

The present trend seems more much hopeful of a final solution of the difficulty; it consists of an attempt on the part of the mathematician to bridge over the gap by supplementing the theoretic solution with a systematic development of the graphical and numerical work that shall be at once economically feasible, yet sufficiently accurate for the problem at hand. Professor Timerding, the editor, is certainly qualified to speak as a mathematician; his experience at the technical school at Brunswick has enabled him to learn the needs of the engineer, hence we can expect the handbook to contain something worth while. The first volume is prepared by Dr. von Sanden, of the University of Göttingen. Both the book and its author developed under the immediate influence of Professor Runge, while the emphasis given to graphical skill shows the effects of Schilling's activity in the organization of the geometric institute.

The first chapter is concerned with arrangement of numerical calculation; it admonishes against the use of detached notes, and urges to keep every step of each calculation in tabulated form; it points out that a numerical error should show up in an unwarranted roughness in the graphical representation. The theory and the use of the slide-rule are developed in great detail. The treatment of this chapter is fairly typical of the whole plan. The theory is presented with sufficient rigor to satisfy the most exacting critic, yet its application to a wide range of numerical processes is so minutely explained that a reader with patience may soon become skilled in its use. Calculating machines are then treated in the same way. The planimeter, integraph, and harmonic analyzer are simply mentioned.

The chapter on Horner's method is almost exactly the same as the corresponding chapter in an American algebra; almost the same can be said about the chapter on interpolation. In mechanical quadrature, the usual treatment of the trapezoidal rule and Simpson's rule is followed by the more comprehensive method of Gauss.

The second half of the book is devoted more particularly to graphical differentiation and integration, as developed in

Professor Runge's Columbia University Lectures. The remainder in Taylor's series, the forms of trigonometric series, and of series in harmonic functions are treated at length. The discussion of systems of simultaneous equations by means of successive elimination, by least squares, and by the method of Gräffe, emphasize the fact that the ordinary expression of a root as the quotient of two determinants has but little value for numerical purposes. The last two chapters are concerned with the graphical and numerical solution of ordinary differential equations of the first and second orders. The book is supplied with a list of references for further study. The table of contents and clearly arranged subject matter make an index unnecessary.

While there was a distinct need for such a book as that of Dr. von Sanden, especially in Germany, it is necessary to look rather more carefully to discover the need of a new treatise on descriptive geometry. But when we recall that the series of books is to form a handbook for the teacher, to supply the missing step between theory and practice, the purpose is easily seen. The book begins with an unusually long discussion of orthogonal projection on a single plane; numerous applications are given, and the reader becomes familiar with this one dominant process. The next chapter of less than twenty pages is devoted to the double projection of rectilinear figures. It contains all the essential features of the methods of descriptive geometry, as applied to straight lines and planes. Then follows a short introduction to axonometry and cavalier projection. The elements of projective geometry, including homology and the cross-ratio theorems of Pappus and of Desargues, are fully treated, and a fairly full metrical theory of conics is added.

The part on plane curves includes tangent, normal, radius of curvature, evolute, with applications to envelopes, roulettes, and cyclic curves. It is more purely geometric and emphasizes the graphical properties more than most recent books on the subject.

Surfaces of revolution are considered almost exclusively from the standpoint of double projection. A chapter of twenty pages is given to the study of quadric surfaces; it includes center, diametral planes, contour, rectilinear generators, circular sections, and stereographic projection. The study of the space curve begins with the simple arc. The projection along a bisecant and along a tangent are clearly explained, also from a point in the osculating plane. Cones and developables, together with their curves of intersection, ruled surfaces and helicoids complete the volume.

The presentation is too concise for a first reading, but the volume is not meant for this purpose. It is rather for the teacher who already knows something of the various methods and wishes to know their mutual relations. At the end of each chapter a generous list of books and monographs is given which add greatly to the value of the book. It is a curious fact that not a single American work is mentioned.

Virgil Snyder.

NOTES.

The December number (volume 16, number 2) of the *Annals of Mathematics* contains the following papers: "A substitute for Duhamel's theorem," by G. A. Bliss; "The points of inflexion of a plane cubic curve," by L. E. Dickson; "Properties of four confocal parabolas whose vertical tangents form a square," by C. M. Herbert; "Some remarks on conformal representation," by T. H. Gronwall; "On the maximum modulus of an analytic function," by T. H. Gronwall; "Note on the simple difference equation," by J. H. M. Wedderburn; "Note on the rank of a symmetrical matrix. II," by J. H. M. Wedderburn.

At the Philadelphia meeting of the American association for the advancement of science Professor W. W. Campbell was elected president, and Dr. L. O. Howard was reëlected permanent secretary for a term of five years. Professor A. O. Leuschner was elected vice-president of Section A. The Association will hold a summer meeting at San Francisco, August 2–7, and a winter meeting at Columbus, Ohio, next December. A convocation week meeting will be held in New York City in 1916–1917.

The Paris academy of sciences announces the following prize problems. The Bordin prize (3,000 fr.) for 1915 for a noteworthy contribution to the theory of curves of constant torsion, in particular of algebraic curves, with special emphasis on rational curves; the grand prize (3,000 fr.) for 1916 for the application of the methods of Poincaré to the integration of linear differential equations of the first order; the Bordin prize (3,000 fr.) for 1917 for the arithmetic theory of non-quadratic forms; the Vaillant prize (4,000 fr.) for 1917 for the determination of all surfaces which can be generated in two ways by the displacement of a rigid curve.

THE following courses in mathematics are being given during the present semester.

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ: Differential calculus, with exercises, five hours; Problems in conformal representation, two hours; Seminar, two hours; Colloquium, two hours.—By Professor G. FROBENIUS: Algebra, four hours; Seminar, two hours.—By Professor F. SCHOTTKY: General theory of functions, four hours; Theory of the potential, four hours; Seminar, two hours.—By Professor J. KNOBLAUCH: Surfaces and space curves, four hours; Problems in mathematical pedagogy, four hours; Selected chapters in elliptic functions, two hours.—By Dr. A. KNOPP: Integral calculus, with exercises, five hours; Theory of infinite series, II, four hours.—By Professor F. R. HELMERT: Method of least squares, one hour.

UNIVERSITY OF FRANKFORT.—By Professor A. SCHOENFLIES: Projective geometry, four hours; Theory of sets, with exercises, two hours; Seminar, two hours.—By Professor E. HELLINGER: Differential equations, with exercises, four hours; Theory of integral equations, three hours.—By Dr. O. SZÀSZ: Theory of numbers, with exercises, three hours; Theory of continued fractions, two hours.—By Professor M. BRENDEL: Celestial mechanics, three hours; Mathematics of insurance, two hours.

UNIVERSITY OF GÖTTINGEN.—By Professor D. HILBERT: Principles of mathematics, four hours; Seminar, two hours.—By Professor E. LANDAU: Differential and integral calculus, II, with exercises, five hours; Seminar, two hours.—By Professor C. RUNGE: Graphical methods, with exercises, six hours; Seminar, two hours.—By Dr. H. v. SANDEN: Mathematical treatment of the natural sciences, three hours; Projective geometry, with exercises, four hours; Theory of mathematical instruments, three hours; Colloquium, two hours.—By Dr. E. HECKE: Curves and surfaces, four hours; Applications of the theory of functions to algebra, two hours.—By Dr. H. BEHRENS: Elementary theory of numbers, four hours; Galois theory of equations, two hours.—By Dr. R. COURANT: Selected chapters of elementary mathematics, three hours; Partial differential equations of mathematical physics, four hours; Selected chapters in the theory of functions, two hours.—By Dr. F. BERNSTEIN: Mathematical statistics and biometrics, two hours; Mathematics of insurance, four hours; Seminar, two hours.

UNIVERSITY OF LEIPZIG.—By Professor O. HÖLDER: Elliptic functions, four hours; Theory of algebraic equations, two hours; Seminar, two hours.—By Professor K. ROHN: Differential geometry, with exercises, five hours; Descriptive geometry, with exercises, four hours.—By Professor G. HERGLOTZ: Differential and integral calculus, with exercises, six hours; Analytic geometry of space, three hours.—By Professor H. BRUNS: Theory of errors, two hours; Practical analysis, two hours.

UNIVERSITY OF STRASSBURG.—By Professor F. SCHUR: Projective geometry, four hours; Foundations of geometry, four hours; Seminar, two hours.—By Professor G. FABER: Algebra, four hours; Partial differential equations, four hours; Seminar, two hours.—By Professor M. SIMON: History of mathematics in the middle ages, four hours.—By Professor J. WELLSTEIN: Differential and integral calculus, with exercises, five hours.—By Professor R. v. MISES: Analytic geometry, four hours; Mechanics of rigid bodies, four hours; Seminar, two hours.—By Professor S. EPSTEIN: Hypergeometric differential equations, four hours.—By Dr. A. SPEISER: Analytic functions and Riemann surfaces, four hours.

DR. H. FALCKENBERG has been appointed docent in mathematics at the technical school at Brunswick.

DR. G. PÓLYA has been appointed docent in mathematics at the technical school of Zürich.

DR. E. ROSATI has been appointed docent in geometry at the University of Pisa.

DR. A. TONOLO has been appointed docent in mathematics at the University of Padua.

DR. A. ARMELLINI has been appointed docent in celestial mechanics at the University of Rome.

MR. A. BERRY has been appointed lecturer in mathematics at the University of Cambridge.

PROFESSOR J. N. HATZIDAKIS, of the University of Athens, has retired from active teaching with the title of professor emeritus.

PROFESSORS S. JOLLES and G. SCHEFFERS, of the technical school of Berlin, have received the title of Geheimer Regierungsrat.

PROFESSORS A. BRILL, of the University of Tübingen, and M. PLANCK, of the University of Berlin, have been elected members of the Accademia dei Lincei of Rome.

DR. G. P. THOMSON has been appointed lecturer in mathematics at Corpus Christi College, Cambridge.

AT the Georgia School of Technology, Mr. D. M. SMITH has been promoted to an assistant professorship of mathematics.

PROFESSOR R. M. BARTON, of the University of New Mexico, has been appointed professor of mathematics in Lombard College.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AUBERT (P.) et PAPELIER (G.). Exercises de géométrie analytique. Paris, Vuibert, 1914. 8vo. 360 pp. Fr. 6.00

BARRAU (J. A.). Ruimtezin en ruimteleer. Groningen, 1913. 8vo. 24 pp.

BERLINER (P. H.). Involutionssysteme in der Ebene des Dreiecks. Braunschweig, Vieweg, 1914. 8vo. 12+212 pp. M. 8.00

COBB (C. W.). Asymptotic development for a certain integral function of zero order. (Diss., University of Michigan.) Norwood, Mass., Norwood Press, 1913. Privately printed. 12mo. 13 pp.

DOWLING (L. E.) and TURNEAURE (F. E.). Analytic geometry. New York, Holt, 1914. 12+266 pp. $1.60

Echols (W. H.). On the roots of a monogenic function inside a closed contour along which the modulus is constant. Charlottesville, 1913.

Franciosi (M.). Varietà triquadratica V_2^8 e sua proiezione da una retta su un S_3. Napoli, tip. r. Accademia delle Science fisiche e matematiche, 1914. 8vo. 22 pp.

Fueter (R.). Die Klassenkörper der komplexen Multiplikation und ihr Einfluss auf die Entwicklung der Zahlentheorie. Bericht zur Feier des 100. Geburtstags Eduard Kummers. Sonderabdruck aus dem 20. Bande des *Jahresberichts der Deutschen Mathematiker-Vereinigung.* Leipzig, Teubner, 1911. 8vo. 47 pp. M. 1.50

Gasiorowski (L.). Ueber die Definitionsgleichungen der endlichen kontinuierlichen Gruppen von Berührungstransformationen in der Ebene. (Diss.) Giessen, 1914.

Goodenough (G. A.). See Townsend (E. J.).

Haag (J.). Cours complet de mathématiques spéciales. Tome I: Algèbre et analyse. Paris, 1913.

Harnack (A.). See Serret (J. A.).

Hero Alexandrinus. See Sass (C.).

Kuschke (C. G. P.). The abelian equations of the 10th degree irreducible in a given domain of rationality. (Diss.) Berkeley (*Univ. Calif. Publ.*), 1914. Royal 8vo. 48 pp. $0.50

Lebon (E.). Savants du jour: Emile Picard. Biographie, bibliographie analytique des écrits. 2me édition, entièrement refondue. Paris, Gauthier-Villars, 1914. 8vo. 8+80 pp. Fr. 7.00

Mangoldt (H. von). Einführung in die höhere Mathematik für Studierende und zum Selbstudium. Band III. Leipzig, Hirzel, 1914. 8vo. 10+485 pp. Geb. M. 14.60

Neumann (C.). Ueber die Dirichletsche Theorie der Fourierschen Reihen. Versuch die Dirichletsche Theorie so umzugestalten, dass sie Auskunft gibt nicht nur uber die Gleichwertigkeit zwischen der gegebenen Funktion und der ihr entsprechenden Fourierschen Reihe, sowie uber die Konvergenz der Reihe, sondern auch uber die Gleichmässigkeit dieser Konvergenz. Leipzig (*Abh. Ges. Wiss.*), 1914. 86 pp. M. 3.00

Papelier (G.). See Aubert (P.).

Poletti (L.). Risultati teorico-pratici di una radicale modificazione del crivello di Eratostene. Parma, 1914. 4to. 47 pp.

Sass (C.). De Heronis Alexandrini quae feruntur definitionibus geometricis. (Diss.) Gryphiswaldensis, 1913.

Scheffers (G.). See Serret (J. A.).

Serret (J. A.). Lehrbuch der Differential- und Integralrechnung. Nach Axel Harnacks Uebersetzung. 3ter Band: Differentialgleichungen und Variationsrechnung. 4te und 5te Auflage, bearbeitet von G. Scheffers. Leipzig, Teubner, 1914. 8vo. 14+735 pp. M. 14.00

Tarnutzer (G.). Ueber die kubischen Nullkurven des linearen Komplexes. Zürich, 1913. 8vo. 51 pp.

Townsend (E. J.) and Goodenough (G. A.). Calculus. New York, Holt, 1914. $2.50

——. Essentials of calculus. New York, Holt, 1914. $2.00

TURNEAURE (F. E.). See DOWLING (L. E.).

WAAGE (E.). Zur Tschebyschefschen Primzahlentheorie. Mitteilungen 1–2. Wien (*Sitzb. Akad.*), 1913–1914. Gr. 8vo. 19+18 pp.

WOLFF (H. C.). Mathematics for agricultural students. New York, McGraw-Hill, 1914. 8vo. 310 pp. $1.50

II. ELEMENTARY MATHEMATICS.

BÔCHER (M.) and GAYLORD (H. D.). Trigonometry with the theory and use of logarithms. New York, Holt, 1914. 12mo. 9+142 pp. Cloth. $1.00

BREWSTER (G. W.). See SANDERSON (F. W.).

BRUCE (W. H.) and CODY (C. C.). Plane and solid geometry. Nashville, Tenn., Southern Publ. Assn., 1913. $1.25. Teacher's edition, 1913. $1.25

BUSCH (F.). See FÉAUX (B.).

CHADSEY (C. E.) and SKINNER (H. M.). Advanced arithmetic. Boston, Atkinson, Mentzer & Co., 1914. $0.55.

——. Elementary arithmetic. Boston, Atkinson, Mentzer & Co., 1914. 14+220 pp. $0.35

——. Intermediate arithmetic. Boston, Atkinson, Mentzer & Co., 1914. $0.45

CODY (C. C.). See BRUCE (W. H.).

FÉAUX (B.). Ebene Trigonometrie und elementare Stereometrie. 9te Auflage, von F. Busch. Paderborn, Schöningh, 1914. 196 pp. Geb. M. 2.20

——. Lehrbuch der elementaren Planimetrie. 11te Auflage, von F. Busch. Paderborn, Schoningh, 1914. 223 pp. Geb. M. 3.00

FELDMAN (D. D.). See HART (C. A.).

FRINK (F. H.). See HALL (A. G.).

GAYLORD (H. D.). See BÔCHER (M.).

GRENVILLE (L. W.). Key to Hall's School algebra. Parts I, II, III. London, Macmillan, 1914. Cloth. 10s.

GUTZMER (A.). Die Tatigkeit des deutschen Ausschusses für den mathematischen und naturwissenschaftlichen Unterricht in den Jahren 1908 bis 1913. Leipzig, Teubner, 1914. Gr. 8vo. 8+482 pp. Geb. M. 12.00

HALL (A. G.) and FRINK (F. H.). Plane trigonometry with trigonometric and logarithmic tables. New York, Holt, 1914. $1.25

HALL (H. S.). See GRENVILLE (L. W.).

HART (C. A.) and FELDMAN (D. D.). Plane and solid geometry. New York, American Book Co., 1912. 12mo. 8+488 pp. Cloth. $1.25

HEIBERG (J. L.). See HERO.

HENKLER (P.) und KÖRNER (K.). Arithmetik und Geometrie für die oberen Klassen. 2ter Teil, 1ter Band von P. Henklers Mathematischem Lehr- und Uebungsbuch fur die Lehrerbildungsanstalten. Berlin, Union Deutsche Verlagsgesellschaft, 1914. 398 pp.

HERO ALEXANDRINUS. Opera quae supersunt omnia. Volume V: Stereometrica et de mesuris. Copiis G. Schmidt usus edidit J. L. Heiberg. (Graece et Germanice.) Leipzig, Teubner, 1914. 8vo. 126+275 pp M. 10.00

KÖRNER (K.). See HENKLER (P.).

KRAUS (K.). Grundriss der Geometrie und des geometrischen Zeichnens fur Lehrerbildungsanstalten. 4te Auflage. Wien, A. Pichlers Witwe, 1914. 245 pp. Geb. Kr. 2.70

KRIMPHOFF (W.). See SCHWERING (K.).

LAWSON (G). A new geometry for schools. Book I: Triangles and. parallelograms. Edinburgh, Chambers, 1914. 8vo. 2s.

LEATHER (H.). Pendlebury's new concrete arithmetic. First to fifth years. London, Bell, 1914. 4+4+4+6+6d.

LESSER (O.). See SCHWAB (K.).

LÖFFLER (E.). See LÖRCHER (O.).

LÖRCHER (O.) und LÖFFLER (E.). Methodischer Leitfaden der Geometrie nebst einer Vorschule der Trigonometrie für hohere Lehranstalten. Stuttgart, Grub, 1914. 201 pp. M 2.80

MILLS (J. F.). See STONE (J. C.).

MILNE (R. M.). Mathematical papers for admission into the Royal military academy and the Royal military college, July, 1914. London, Macmillan, 1914. 1s.

MILNE (W. J.). New York state arithmetic. Books 1–2. New York, American Book Co., 1914. $0.40+0.40

MÜLLER (C. H.). See SCHWAB (K.).

PAEHLER (F.). Rechenbuch für hohere Lehranstalten. 1tes Heft: für Sexta; 2tes Heft: fur Quinta; 3tes Heft: für Quarta. Frankfurt a. M., Kesselbring, 1914. 91+87+156 pp.

PATERSON (W. E.). Five-figure logarithmic and trigonometrical tables. Oxford, Clarendon Press, 1914. Cr. 8vo. 28 pp. 1s.

RENAUT (C.). Nouvelle table pour le calcul des dixièmes, des neuvièmes, des huitièmes, des septièmes, échus sur contributions, pour tous les nombres dep. 0 jusqu'à 1019. Paris, 1913. Fr. 4.00

SANDERSON (F. W.) and BREWSTER (G. W.). A geometry for schools. Cambridge, University Press, 1914. Cr. 8vo. With answers, 10+336 pp.; also without answers. Cloth. 2s. 6d.

SCHMEHL (C.). Rechenbuch für hohere Lehranstalten. 12te Auflage. 2 Teile. Giessen, E. Roth, 1914. 236+267 pp. Geb. M. 3.70

SCHMIDT (G.). See HERO.

SCHUBERT (H.). Niedere Analysis. 2ter Teil: Funktionen, Reihen, Gleichungen. (Sammlung Schubert, Band XLV.) 2te Auflage. Leipzig, Goschen, 1911. 12mo. 4+216 pp. Cloth. M. 3.80

SCHWAB (K.) und LESSER (O.). Mathematisches Unterrichtswerk. Leipzig, Freytag, 1914. Lehr- und Uebungsbuch fur den Unterricht in der Arithmetik und Algebra von O. Lesser; 1ter Teil, Ausgabe B für Gymnasien, besorgt von C. H. Muller. 1te Auflage. 162 pp. Lehr- und Uebungsbuch der Geometrie von K. Schwab; 1ter Teil, Ausgabe A für Realanstalten. 4te Auflage. 281 pp. Geb. Dasselbe, 2ter Teil, Ausgabe A fur Realanstalten. 3te Auflage. 138 pp. Geb. M. 2.00+3.50+1.90

SCHWERING (K.) und KRIMPHOFF (W.). Ebene Geometrie. 8te Auflage. Freiburg i. Br., Herder, 1914. 143 pp. Geb. M. 2.50

SKINNER (H. M.). See CHADSEY (C. E.).

SLAUGHT (H. E.). See WILCZYNSKI (E. J.).

SMITH (D. E.). See WENTWORTH (G.).

STONE (J. C.) and MILLS (J. F.). Arithmetics, complete. 2d edition. Boston, Sanborn, 1914. $0.60

TRENARD (H.). Arithmétique et algèbre, 1er année. 3e édition. Paris, 1913. Fr. 1.60

WENTWORTH (G.) and SMITH (D. E.). Vocational algebra. Boston, Ginn, 1911. 12mo. 6+88 pp. Cloth. $0.50

WILCZYNSKI (E. J.). Logarithmic and trigonometric tables. Edited by H. E. Slaught. Boston, Allyn and Bacon, 1914. 8vo. 20+97 pp. Cloth. $1.00

——. Plane trigonometry and applications. Edited by H. E. Slaught. Boston, Allyn and Bacon, 1914. 8vo. 11+265 pp. Cloth. $1.25

YOUNG (J. W. A.) and JACKSON (L. L.). High school algebra. New York, Appleton, 1913. $1.15

III. APPLIED MATHEMATICS.

AMERICAN SCHOOL OF CORRESPONDENCE. Practical mathematics, parts 1–3: instruction paper. 3 volumes. Chicago, Armour Institute, 1911. $1.50

APPELL (P.). See COFFIN (J. G.).

AUWERS (A.). Bearbeitung der Bradley'schen Beobachtungen an den alten Meridianinstrumenten der Greenwicher Sternwarte. Band III: Der Sternkatalog. Leipzig, 1914. Gr. 4to. 90 pp. M. 30.00

BATARDON (L.). Compatibilité commerciale, les procédés modernes, la méthode centralisatrice. 2e édition. Paris, Dunod et Pinat, 1914. 8vo. 8+188 pp. Fr. 3.50

BOLZANO (B.). B. Bolzanos Wissenschaftslehre, neu herausgegeben von A. Höfler. 1ter Band. Leipzig, Meiner, 1914. 16+571 pp.

BONNAR (W.). Mathematical laws of psychic phenomena. Buffalo, N. Y., Bonnar, 1912. 220 pp. $5.00

BRILL (A.). Das Relativitatsprinzip. Eine Einführung in die Theorie. 2te Auflage. Leipzig, Teubner, 1914. 8vo. 34 pp. M. 1.20

CHOMÉ (F.). Géométrie descriptive de l'école militaire. Livre II à l'usage des élèves de l'enseignement supérieur. 3e édition revue, corrigée et augmentée. Paris, Gauthier-Villars, 1914. 7+316 pp. Fr. 15.00

CHOWLSON (O. D.). Traité de physique. Traduit sur les éditions russe et allemande par E. Davaux, avec notes sur la physique théorique par E. et F. Cosserat (5 volumes). Volume V, fascicule 1: Champ magnétique variable. Paris, 1914. Pp. 1–266. Fr. 9.50

COFFIN (J. G.). Calcul vectoriel avec application aux mathématiques et à la physique. Traduction et notation française par Alex. Véronnet. Lettre au traducteur par Paul Appell. Paris, Gauthier-Villars, 1914. 8vo. 18+212 pp. Fr. 7.50

COSSERAT (E. et F.). See CHOWLSON (O. D.).

DAVAUX (E.). See CHOWLSON (O. D.).

DE VILLAMIL (R.). Motion of liquids. London, Spon, 1914. 8vo. 14+210 pp. 7s. 6d.

DODD (E. L.). Error-risk of the median compared with that of the arithmetic mean. (*Bull. Univ. Texas*, no. 323; Scientific Series, no. 27.) Austin, 1914. 11 pp.

EDDINGTON (A. S.). Stellar movements and the structure of the universe. London, Macmillan, 1914. 8vo. 12+266 pp. 6s.

FOERSTER (W.). Kalenderwesen und Kalenderreform. Braunschweig, 1914. 8vo. 49 pp. M. 1.60

GALLE (A.). Das Geoid im Harz. (*Veröffentlichung des k. Preussischen Geodätischen Institutes.* Neue Folge, Nr. 61.) Berlin, Reichsdruckerei, 1914. 4to. 102 pp.

GRIMSEHL (E.). Lehrbuch der Physik. 3te vermehrte Auflage. Band I: Mechanik, Akustik und Optik. Leipzig, Teubner, 1914. Gr. 8vo. 12+966 pp. M. 16.00

HAESE (A.). See TROST (W.).

HELMHOLTZ (H. VON). Vorlesungen über theoretische Physik. Herausgegeben von A. Konig, O. Krigar-Menzel und C. Runge. Band V: Vorlesungen uber elektromagnetische Theorie des Lichts (1897). 2ter unveränderter Abdruck. Leipzig, 1914. 8vo. 12+370 pp. M. 14.00

HÖFLER (A.). See BOLZANO (B.).

KLEIN (F.). See SCHOUTEN (J. A.).

KÖNIG (A.). See HELMHOLTZ (H. VON).

KRIGAR-MENZEL (O.). See HELMHOLTZ (H. VON).

LANDI (D.). Disegno di projezioni ortogonali. 2a edizione. Milano, 1914. 12mo. 8+152 pp. L. 2.00

MANAIRA (A.). Intorno alla identità fra il problema della composizione delle forze applicate ad un corpo e quello della composizione delle rotazioni: nota. Caltanisetta, Castaldi, 1914. 8vo. 5 pp.

MARSH (H. W.). Constructive text-book of practical mathematics. Volume III: Technical geometry. New York, Wiley, 1914. 12mo. 14+244 pp. Cloth. $1.25

MELDAU (H.). See SCHILLING (C.).

MICHAELIS (L.). The dynamics of surfaces. An introduction to the study of biological surface phenomena. Translated by W. H. Perkins. London, Spon, 1914. 8vo. 118 pp. 4s.

NEUBECKER (W.). Mensuration for sheet metal workers. New York, Williams, 1914. 51 pp. $0.75

PASCHOUD (M.). Sur l'application de la méthode de Walther Ritz à l'étude de l'équilibrium élastique d'une plaque carrée mince. (Thèse.) Paris, Gauthier-Villars, 1914. 4to. 4+54 pp. Fr. 4.00

PERKINS (W. H.). See MICHAELIS (L.).

PIRANI (M. VON). Graphische Darstellung in Wissenschaft und Technik. Leipzig Göschen, 1914. 12mo. 126 pp. M. 0.90

POLARA (V.). L'elettrodinamica di Maxwell e la moderna elettronica. Catania, Operaia, Nicolos e Giuffrida, 1914. 8vo. 2+112 pp.

PRZYBYLLOK (E.). Polhohen-Schwankungen. Braunschweig, 1914. 8vo. 41 pp. M. 1.60

RAPER (H.). The practice of navigation and nautical astronomy. 20th edition. London, J. D. Potter, 1914. 25+934 pp.

REDGROVE (H. S.). Experimental mensuration. An elementary textbook of inductive geometry. New York Van Nostrand, 1914. 346 pp. $1.25

RIEBESELL (P.). Photogrammetrie in der Schule. Programm, Hamburg. Leipzig, Quelle & Meyer, 1914. 22 pp. M. 0.8

RUNGE (C.). See HELMHOLTZ (H. VON).

SCHILLING (C.) und MELDAU (H.). Der mathematische Unterricht an den deutschen Navigationsschulen. (Abhandlungen über den mathematischen Unterricht in Deutschland, Band IV, Heft 4.) Leipzig, Teubner, 1911. 8vo. 6+82 pp. M. 2.00

SCHOUTEN (J. A.). Grundlagen der Vektor- und Affinoranalysis. Mit Einfuhrungswort von F. Klein. Leipzig, 1914. 8vo. 8+266 pp. M. 11.00

STANLEY (W. F.). Surveying and levelling instruments theoretically and practically described. Revised by H. T. Tallack. For construction, qualities, selection, preservation, adjustments and uses; with other apparatus used by civil engineers and surveyors in the field. 4th edition. London, Spon, 1914. 606 pp. 7s. 6d.

TALLACK (H. T.). See STANLEY (W. F.).

TIMERDING (H. E.). See TROST (W.).

TROST (W.). Die mathematischen Facher an den niederen gewerblichen Lehranstalten in Deutschland. Mit einem Vorwort von H. E. Timerding und einem Anhang von A. Haese. (Abhandlungen über den mathematischen Unterricht in Deutschland, Band IV, Heft 5.) Leipzig, Teubner, 1913. 8vo. 6+150 pp. M. 4.00

VAN DE SANDE BAKHUYZEN (H. G.). Bericht uber das Entstehen und die Entwicklung der internationalen Erdmessung 1862–1912. Leiden, 1913. 4to. 11 pp.

VÉRONNET (A.). See COFFIN (J. G.).

YULE (G. U.). An introduction to the theory of statistics. 2d edition, revised. London, Griffin, 1912. 8vo. 15+381 pp. 10s. 6d.

THE TWENTY-FIRST ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-first annual meeting of the Society was held in New York City on Friday and Saturday, January 1–2, 1915. The attendance at the four sessions included the following ninety-five members:

Mr. J. W. Alexander, II, Dr. E. S. Allen, Professor G. N. Armstrong, Professor C. S. Atchison, Professor Clara L. Bacon, Mr. A. A. Bennett, Professor G. D. Birkhoff, Professor Joseph Bowden, Professor J. W. Bradshaw, Professor E. W. Brown, Dr. T. H. Brown, Dr. R. W. Burgess, Professor B. H. Camp, Professor C. W. Cobb, Professor A. B. Coble, Dr. Emily Coddington, Professor A. Cohen, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Professor Elizabeth B. Cowley, Professor G. H. Cresse, Professor L. P. Eisenhart, Professor G. C. Evans, Mr. G. W. Evans, Professor F. C. Ferry, Professor H. B. Fine, Dr. C. A. Fischer, Professor W. B. Fite, Professor T. M. Focke, Professor A. B. Frizell, Professor O. E. Glenn, Professor J. W. Glover, Dr. G. M. Green, Professor T. H. Gronwall, Professor C. C. Grove, Professor J. G. Hardy, Professor C. N. Haskins, Professor H. E. Hawkes, Professor E. R. Hedrick, Dr. A. A. Himowich, Professor T. F. Holgate, Professor L. A. Howland, Professor E. V. Huntington, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. J. K. Lamond, Professor O. C. Lester, Professor Florence P. Lewis, Mr. P. H. Linehan, Professor W. R. Longley, Professor A. C. Lunn, Professor James Maclay, Professor Mansfield Merriman, Dr. E. J. Miles, Professor H. H. Mitchell, Dr. R. L. Moore, Dr. F. M. Morgan, Professor Frank Morley, Professor F. R. Moulton, Mr. G. W. Mullins, Professor G. D. Olds, Professor W. F. Osgood, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor A. D. Pitcher, Professor Arthur Ranum, Dr. H. W. Reddick, Professor R. G. D. Richardson, Professor H. L. Rietz, Dr. R. B. Robbins, Professor E. D. Roe, Jr., Dr. Joseph Rosenbaum, Dr. Caroline E. Seely, Professor L. P. Siceloff, Dr. L. L. Silverman, Professor H. E. Slaught, Professor P. F. Smith, Professor Virgil Snyder, Professor K. D. Swartzel, Professor Elijah Swift, Professor Evan Thomas, Professor H. W. Tyler, Mr. H. S. Vandiver, Professor E. B. Van Vleck, Professor Oswald Veblen, Mr. J. N. Vedder, Mr. H. E. Webb, Professor C. J. West, Professor H. S. White, Professor E. E. Whitford, Professor J. W. Young.

The President of the Society, Professor E. B. Van Vleck, occupied the chair at the opening session, being relieved by Vice-President L. P. Eisenhart and after the annual election by the President-elect, Professor E. W. Brown, and Vice-President Oswald Veblen. The council announced the election of the following persons to membership in the Society: Dr. Florence E. Allen, University of Wisconsin; Dr. Nathan Altshiller, University of Washington; Dr. D. F. Barrow, University of Texas; Dr. R. B. Robbins, Sheffield Scientific School; Mr. C. H. Yeaton, University of Chicago. Fifteen applications for membership in the Society were received.

The total membership of the Society is now 709, including 69 life members. The total attendance of members at all meetings of the past year was 428; the number of papers read was 192. The number of members attending at least one meeting during the year was 255. At the annual election 210 votes were cast. The Treasurer's report shows a balance of $9,461.75, including the life membership fund of $5,198.85. Sales of the Society's publications during the year amounted to $1,843.67. The Library now contains about 5,100 bound volumes.

The annual meeting this year was especially marked as the occasion of the delivery of President Van Vleck's retiring address, the subject of which was "The rôle of the point set theory in geometry and dynamics." The address was given on Friday afternoon before an audience of nearly one hundred, a considerable number having come on from the West and from the American Association meeting at Philadelphia.

Another special feature of the meeting, on the social side, was the annual dinner and smoker held at the Yale Club on Friday evening with an attendance of seventy members.

At the annual election, which closed on Saturday morning, the following officers and other members of the Council were chosen:

President,	Professor E. W. BROWN.
Vice-Presidents,	Professor F. R. MOULTON, Professor OSWALD VEBLEN.

Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.
Librarian,	Professor D. E. SMITH.

Committee of Publication,
Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to Serve until December, 1917,

Professor G. D. BIRKHOFF,	Professor R. G. D. RICHARDSON,
Professor O. E. GLENN,	Professor W. H. ROEVER.

Committees were appointed to consider the question of holding a colloquium in connection with the summer meeting at San Francisco, and the possible relations of the Society to the field now covered by the *American Mathematical Monthly*. A committee is also considering matters relating to the teaching of mathematics in the schools.

The following papers were read at this meeting:

(1) Professor L. P. EISENHART: "Transformations of surfaces Ω."

(2) Dr. L. L. SILVERMAN: "On the notion of summability for the limit of a function of a continuous variable."

(3) Professor A. B. COBLE: "A configuration in finite geometry."

(4) Professor A. B. COBLE: "The elliptic norm curve in S_4."

(5) Professor J. E. ROWE: "The symbolic and actual form of certain combinants of two binary n-ics."

(6) Professor ARTHUR RANUM: "On the differential geometry of the cyclic (circled) surfaces."

(7) Professor A. B. FRIZELL: "An enumeration of integral algebraic polynomials."

(8) Dr. DUNHAM JACKSON: "Expansion problems with irregular boundary conditions."

(9) Dr. G. M. GREEN: "Hypersurfaces and families of curves defined by solutions of a partial differential equation of the second order."

(10) Dr. CAROLINE E. SEELY: "Certain non-linear integral equations."

(11) President E. B. VAN VLECK, Presidential address: "The rôle of the point set theory in geometry and dynamics."

(12) Professor W. F. OSGOOD: "On the division of space of n dimensions by a simple closed manifold."

(13) Professor G. D. BIRKHOFF: "The functions of several variables defined by linear difference and differential equations."

(14) Professor G. D. BIRKHOFF: "Note on the reducibility of maps."

(15) Professors VIRGIL SNYDER and F. R. SHARPE: "Certain quartic surfaces belonging to infinite discontinuous cremonian groups."

(16) Mr. H. S. VANDIVER: "A property of cyclotomic integers and its relation to Fermat's last theorem."

(17) Professor J. L. COOLIDGE: "Circular transformations and complex space."

(18) Professor G. A. MILLER: "Note on several theorems due to A. Capelli."

(19) Professor EDWARD KASNER: "The generalized concept of differential element."

(20) Professor F. N. COLE: "Note on solvable quintics."

(21) Professor G. C. EVANS: "Note on the variation of a function depending on all the values of another function."

(22) Professor E. V. HUNTINGTON: "A set of postulates for elementary dynamics" (preliminary communication).

(23) Professor F. R. MOULTON: "The solution of an infinite system of implicit functions, with an application to Hill's lunar theory."

(24) Professor C. N. HASKINS: "On the roots of the incomplete gamma function."

(25) Dr. W. C. GRAUSTEIN: "On the geodesics and geodesic circles on a developable surface."

(26) Dr. D. F. BARROW: "Oriented circles in space."

(27) Professor JAMES MACLAY: "A transformation of polynomials relative to the exponents."

(28) Mr. J. W. ALEXANDER, II: "A method for resolving the singularities of algebraic manifolds."

(29) Professor T. H. GRONWALL: "On the summation method of de la Vallée-Poussin."

(30) Professor T. H. GRONWALL: "An integral equation of the Volterra type."

(31) Professor T. H. GRONWALL: "On the distortion in conformal representation."

The papers of Professor Rowe, Dr. Seely, Mr. Vandiver, Professor Miller, Professor Cole, Dr. Graustein, Dr. Barrow, and the last two papers of Professor Gronwall were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In the October number of the *Transactions* Professor Eisenhart showed that by an adaptation of the theorem of Moutard for differential equations of the Laplace type with equal invariants it is possible to transform a surface S, referred to a conjugate system with equal point invariants, into a surface S_1 upon which the parametric curves are of the same sort. This transformation, which he called a transformation K, is such that for the congruence of lines joining corresponding points on S and S_1 the developables cut these surfaces in the parametric system and the focal points on any line are harmonic with respect to the points of S and S_1 on the line. In the present memoir he considers the particular case where this congruence is normal. Surfaces orthogonal to such a congruence have been considered by Demoulin, who called them surfaces Ω. A surface S which admits a transform S_1 such that the congruence is normal is characterized by the property that the conjugate system with equal invariants is 2, 0, to use the notation of Guichard. When such a surface is given, the determination of the normal congruence is a question of quadratures. With the aid of the theorem of permutability of transformations K applied to these surfaces S, it is possible to establish transformations A of surfaces Ω into surfaces Ω such that a surface and a transform are the envelope of a two-parameter family of spheres with the lines of curvature corresponding on the two surfaces. These transformations also admit a theorem of permutability.

Isothermic surfaces are surfaces Ω, and the transformations A which give rise to new isothermic surfaces are the transformations D_m, discovered by Darboux and studied at length by Bianchi. Surfaces with isothermal spherical representation of their lines of curvature are surfaces Ω with one of the surfaces S at infinity, and consequently require special study. When the surfaces resulting from such a surface by transformations A are surfaces of the same sort, the transformations are the same as those established by the author some years ago from a different point of view.

2. A sequence of numbers may be regarded as a function $u(n)$ of a discrete variable n. Toeplitz has defined a related sequence

$$v(n) = \sum_{i=1}^{i=n} k(n, i)u(i),$$

where $k(n, i)$ is a function of two discrete variables, and has obtained conditions on $k(n, i)$ which are necessary and sufficient that the existence of the limit of $u(n)$ imply the existence of the limit of $v(n)$ and the equality of the two limits. Replacing the discrete variables by continuous variables and the sign of summation by an integral sign, Dr. Silverman considers the related functions $u(x)$ and

$$v(x) = \int_a^x k(x, y)u(y)dy$$

and obtains conditions on $k(x, y)$ which are necessary and sufficient that the existence of the limit of $u(x)$ imply the existence of the limit of $v(x)$ and the equality of the two limits. The theorem obtained applies in particular to improper integrals.

3. For $p = 1, 2, 3$ there are sets of even theta functions in p variables which are related to a set of $2p + 2$ points in a projective space S_p. The purpose of Professor Coble's first paper is to show that, in the finite geometry associated with the functions, there exist for any value of p configurations which are in tactical correspondence with such a set of points.

4. Among all the elliptic norm curves, that of order five in S_4 is peculiar in that the quadrics which contain the curve serve to define projectively a Cremona transformation. This transformation is studied in Professor Coble's second paper as an aid to the location of the point sets mentioned in the first paper.

5. The important rôle played by the combinants of two binary forms in the theory of rational plane curves is dependent upon the fact that if they are expressed in *combinant form* (i. e., in terms of the two-rowed determinants of the matrix of coefficients of the two binary forms) an easy translation scheme makes it possible to transform these combinants into the equations of covariant curves of rational plane curves. The purpose of Professor Rowe's paper is to derive a series of combinants of two binary n-ics which are not only

in combinant form, but which may be expressed as determinants of comparatively low order whose constituents are functions of the two-rowed determinants of the matrix of coefficients of the two binary forms; also, it is shown how these may be represented symbolically in such a way that the actual expression for a combinant of any two particular n-ics may be written out with facility. Possibly the simplest combinant of this type is the one whose vanishing is the necessary and sufficient condition that two binary n-ics have polar cubics with two common factors.

6. In the study of cyclic surfaces much attention has been given in the past to certain very special kinds, for instance to anallagmatic surfaces, including cyclides. In the theory of the general cyclic surface, very little progress has been made, and the few results obtained have been confined almost exclusively to metrical and conformal properties. The more fundamental descriptive properties, which are invariant under the 24-parameter group of sphere transformations, have hardly been touched. They form the subject of the present paper by Professor Ranum, which is based on two earlier papers on the projective differential geometry of n-space, one published in the *Annali* in 1912 and the other soon to appear in the *American Journal.* The results are obtained by employing the well-known correspondence between the projective line geometry of 4-space and the descriptive circle geometry of 3-space. It is shown that from this viewpoint cyclic surfaces naturally fall into eleven classes, of which four are annular and seven non-annular. Among the striking results is the fact that there exists on every non-annular surface (excepting only those which are anallagmatic) a certain fundamental pair of curves, whose importance for the surface is somewhat similar to that of the edge of regression for a developable surface.

7. By arranging the natural numbers as products of powers of primes, Professor Frizell makes an actual enumeration of the algebraic equations in which all coefficients are positive integers. Similarly the equations in which negative integers occur as coefficients are put into one-to-one correspondence with the set of positive fractions in their lowest terms. Then the two sets of equations may be assigned to the even and odd numbers respectively.

8. In most of the expansion problems which have been studied in detail, in connection with ordinary linear differential equations, the boundary conditions are of the sort which Birkhoff calls *regular*. A notable exception occurs in the case of certain series obtained by Liouville (*Journal de Mathématiques*, volume 3) from differential equations of order higher than the second, with boundary conditions of a particular type. It is readily seen that these conditions are irregular, and consequently the question of the convergence of the series, which was left open by Liouville, also falls outside the scope of Birkhoff's general treatment. Dr. Jackson shows that the difference between this case and those usually studied is an essential one; expansions which would be uniformly convergent, if the analogy of the ordinary theory were sustained, are found actually to diverge with great rapidity. The difference is explained by the fact that the characteristic functions, instead of being essentially of the nature of trigonometric functions, as in the familiar cases, involve real exponential factors which fundamentally affect their properties. The analysis is applicable not only to Liouville's boundary conditions, but also to a considerably more general class of conditions in which these are included.

9. Segre has pointed out that on a surface (two-spread) defined by solutions of a partial differential equation of the second order in any space of higher dimensionality than three, there exists one and only one conjugate net of curves, in the sense that the tangents to curves of one family of the net, at the points where these curves meet a fixed curve of the other family, form a developable surface. It is well known also that with every one-parameter family of curves on a surface in three-space may be associated a unique conjugate family. Dr. Green's paper has two purposes in view: first, to study a surface in a space of n dimensions referred to a conjugate net of parameter curves; and second, to provide a substitute for conjugacy on a surface when $n > 3$, in order to be able to associate with a one-parameter family of curves on such a surface a second family uniquely determined by the first. The quasi-conjugacy defined in the paper is a reciprocal relation, and Segre's unique conjugate net plays the same rôle as does the asymptotic net on a surface in ordinary space, in the sense that each family of the conjugate net is quasi-

conjugate to itself. The analysis of the paper consists in a study not of a single partial differential equation of the form

$$Ay_{uu} + By_{uv} + Cy_{vv} + Dy_u + Ey_v + Fy = 0,$$

but of a completely integrable system of two equations of which the above is one. Wilczynski's methods are used in obtaining the invariants and covariants of the configurations considered.

10. The non-linear integral equations discussed in Miss Seely's paper are of the type susceptible of treatment by Volterra's algebra of permutable and non-permutable functions. Volterra has proved that the binomial quadratic

$$\int_a^b \xi(s, \tau)\xi(\tau, t)d\tau + 2\xi(s, t) + \lambda k(s, t) = 0$$

is satisfied by a certain power series in λ within its circle of convergence, the coefficients being symbolic powers of the known kernel $k(st)$ except for constant factors. Miss Seely proves that the equation is also satisfied by the analytic prolongation of this series at every non-singular point in the λ plane, and shows that the singular points are identical with the characteristic constants of the kernel $k(st)$ as given by the Fredholm theory. If λ_h is a pole of order $m > 1$ of the Fredholm resolvent kernel $k(st, \lambda)$, it is an infinity of order $(2m - 3)/2$ of the solution of the binomial quadratic; if $m = 1$, it is a branch point of first order. An expression is given for the solution valid at every point in its star. Extension is made to binomial integral equations of nth degree, and to the general quadratic and cubic, and some discussion is given of the question of the existence of solutions at the singular points.

11. President Van Vleck's retiring address will appear in full in the BULLETIN for April.

12. Let a simple closed surface in space of three dimensions, which is made up of a finite number of pieces of analytic surfaces meeting one another along a finite number of analytic curves, be denoted as an analytic polyhedron. The corresponding $(n - 1)$-dimensional manifold in space of n dimensions shall be called an analytic hyperpolyhedron. Professor

Osgood proves the theorem that such a manifold divides the space in which it lies into an interior and an exterior region. The method consists in slicing by a family of parallel planes or hyperplanes and considering the regions into which one of them is divided.

13. Professor Birkhoff's first paper contains an extension of the methods of an earlier paper* to the treatment of certain classes of functions of several complex variables, and will appear in the *Acta Mathematica.*

14. In this paper Professor Birkhoff carries further the principle of "reducibility of maps" employed earlier by him,† and thus limits still further the form of "completely reduced" maps. The importance of the notion of reducibility is that it appears probably to afford an adequate although extremely complex method for the solution of the four-color problem.

This paper will be offered to the *American Journal of Mathematics.*

15. The paper of Professors Snyder and Sharpe has for its purpose the establishment of two theorems:

Theorem I. The quartic subjected to the single condition of passing through a non-hyperelliptic sextic curve of genus three is invariant under an infinite discontinuous group of birational transformations.

Theorem II. The transformations of the infinite discontinuous group under which the most general quartic surface passing through a sextic curve of genus two remains invariant can be expressed in terms of cremonian transformations.

In connection with the first theorem the equation of the surface is derived and the equations of the transformations are determined; it is shown that the transformations are cremonian and non-involutorial and that no transformations exist other than those obtained. It is believed that this surface is the first illustration of one which possesses an infinite discontinuous group, but contains neither a pencil of elliptic curves nor a net of hyperelliptic curves of genus two. The equation of the surface (Fano surface) mentioned in the second theorem is found, and also the equations of two involutorial space transformations, which generate the infinite group.

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* "The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations," *Proceedings of the American Academy of Arts and Sciences*, vol. 49 (1913), pp. 521–568.

† *American Journal of Mathematics*, vol. 35 (1913), pp. 115–128.

16. By extending some methods due to Kummer (Berlin *Sitzungsberichte,* 1857) Mr. Vandiver is enabled to set up a relation from which the criteria of Kummer (Mirimanoff, *Crelle,* volume 128), Legendre, and Furtwängler (*Wiener Sitzungsberichte,* 1912, page 589) in reference to Fermat's last theorem may be immediately deduced.

17. In the *Mathematische Annalen,* volume 22 (1883), Stéphanos establishes a one-to-one correspondence between the projectivities of the binary domain and the points of a projective S_3. In Professor Coolidge's paper these formulas are adapted to establishing a one-to-one correspondence between the points of a complex projective S_3 and the real circular transformations of the Gauss plane. The correspondence is used on the one hand to give real representation of various loci in complex space, and on the other to discover new theorems concerning circular transformations.

18. The useful theorem that every subgroup of a group of order p^m, p being any prime number, is invariant under operators of the group which are not contained in this subgroup is commonly supposed to be due to G. Frobenius and to W. Burnside. Cf. "Encyclopédie des Sciences mathématiques" tome I, volume 1, page 608; W. Burnside, "Theory of Groups of Finite Order," 1911, page 122. As a matter of fact, this theorem is evidently a special case of a theorem which was proved eleven years earlier by A. Capelli, and which may be stated as follows: A necessary and sufficient condition that every subgroup of a group G is transformed into itself by at least one operator which is not contained in this subgroup is that G contains no more than one Sylow subgroup of any given order.*

This theorem is clearly equivalent to the theorem that a necessary and sufficient condition that every subgroup of a group G appears in at least one of its possible series of com-

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*A. Capelli, *Atti R. Accademia dei Lincei, Memorie,* (3), vol. 19 (1884), p. 272.

position is that G is the direct product of its Sylow subgroups; which the second edition of Pascal's Repertorium der höheren Mathematik, volume 1, 1910, page 199, accredits to W. Burnside. The object of Professor Miller's paper is to direct attention to this and to other theorems proved by A. Capelli, and to extend one of these theorems. In particular, the operators of any composite group G are arranged in the usual rectangular form with respect to an invariant subgroup H as the first row, and the operators of the first column are so selected as to generate the smallest possible group. The cross-cut of this group and H cannot contain more than one operator, or more than one subgroup, in any complete set of conjugates when this is also a complete set of conjugates under G. In particular, this cross-cut is the direct product of its Sylow subgroups. The smallest group above mentioned must be G itself whenever G is cyclic and H does not include any Sylow subgroup of G.

19. An ordinary differential element in the plane may be represented by a point and a parabolic curve of the nth order passing through it. An infinite power series in x, convergent or divergent, Professor Kasner regards as defining a differential element of infinite order. This may be pictured most concretely, in general, by the broken line running from the given point to the successive centers of curvature. If further we allow series with fractional exponents, we obtain, in addition to regular elements, new types of irregular elements. According to the nature of the coefficients in the series, the element may be real or imaginary. In earlier papers the author has classified elements with respect to the conformal group. The present paper gives the classification of all elements (real or imaginary, regular or irregular, convergent or divergent) in metric geometry.

20. Professor Cole obtains a set of very simple resolvent equations for the algebraic solution of quintics of the form $x^5 + \alpha x + \beta = 0$ when such solution is possible. This paper will appear in a later number of the BULLETIN.

21. In order to express the variation of a function depending continuously on all the values of another function between a and b, in terms of the integral of the derivative of the de-

pendent function with respect to the functional argument multiplied by the variation of that argument, Volterra finds it necessary to make use of four postulates, two of which are statements of uniform conditions. In the present paper Professor Evans seeks to reduce the number of these assumptions. He proves that if the functional derivative exists for every point of a continuous curve $\varphi(x)$ between $x = \xi_1$ and $x = \xi_2$, and takes the values α at ξ_1 and β at ξ_2, then between ξ_1 and ξ_2 it takes all the values between α and β. If the functional derivative of the function F exists for all values of ξ between $x = a$ and $x = b$, and for all functions φ in a certain region, being continuous with respect to φ in that region, and if F vanishes for φ_1 and φ_2, $\varphi_2 - \varphi_1$ being everywhere a positive quantity, then the derivative of F vanishes at some point of one of the curves of the pencil $(\varphi_1\varphi_2)$. If the derivative is continuous in regard to its functional argument in the neighborhood of $\varphi(x)$, and finite, and integrable along $\varphi(x)$ in regard to its other argument ξ from a to b, then the first variation may be written in the desired form. Moreover if in addition, the derivative is continuous in regard to ξ, so that two of Volterra's postulates are established, the other two follow.

22. Professor Huntington's paper contains a tentative set of postulates for the composition and resolution of forces, expressed in terms of the following fundamental concepts: (1) a class of elements $a, b, c, \cdots$, to be called forces, or vectors; (2) a relation $a \uparrow b$, to be called like-directedness; a relation $a \equiv b$, to be called congruence; and a rule of combination $a \oplus b = c$, to be called vector addition.

23. The infinite system of equations considered by Professor Moulton may be written in the form

$$x_1 = f_1(x_1, x_2, \cdots; m) = f_1^{(0)} + f_1^{(2)} + \cdots,$$
$$x_2 = f_2(x_1, x_2, \cdots; m) = f_2^{(0)} + f_2^{(2)} + \cdots,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

where m is an arbitrary parameter, $x_1, x_2, \cdots$ an infinite system of dependent variables, $f_i^{(j)}$ the totality of terms of f_i which are homogeneous of degree j in $x_1, x_2, \cdots$. One set of hypotheses under which the conclusion follows is:

(*a*) The $f_i^{(0)}$ vanish for $m = 0$.
(*b*) The $f_i^{(0)}$ and the coefficients of the $f_i^{(j)}$ are regular analytic functions of m for $|m| \leqq \rho$.
(*c*) There exists a finite constant M_i and a series $c_1x_1+c_2x_2 + \cdots = s$, where the c_i are regular functions of m for $|m| \leqq \rho$, such that s converges if $|m| < \rho$, $|x_i| < r_i$ and $M_i s^j$ dominates $f_i^{(j)}$.
(*d*) $M_i \leqq r_i$ if $|m| \leqq \rho' \leqq \rho$, $|x_i| \leqq r'_r \leqq r_i$.
Under these conditions there exist series of the form

$$x_i = a_i^{(1)}m + a_i^{(2)}m^2 + \cdots$$

which satisfy the original equations identically in m and converge if $|m| < \sigma > 0$.

By an application of these results, suitably modified, the expansions as power series in m of the Fourier coefficients of the expressions for the coordinates in Hill's variational orbit can be shown to converge for $|m|$ sufficiently small. This ocmpletes the proof of the convergence, in sufficiently restricted domains, of all the series Hill employed in his lunar theory.

24. The location of the real roots of the function $P(x)$ defined by

$$P(x) \equiv \frac{\Sigma(-1)^s}{s!}\frac{1}{x+s}$$

has been studied by Bouquet who found that in each of the intervals $-2n < x < -2n + \frac{1}{2}$, $-2n + \frac{1}{2} < x < -2n + 1$ there lies *at least* one root, while outside these intervals there are no real roots. Professor Haskins shows by an application of the theorem of Budan-Fourier that in each of these intervals there is *exactly* one root.

25. If $x_i = x_i(s)$, $(i = 1, 2, 3)$, is the parametric representation (in terms of the arc) of an arbitrary but fixed geodesic of a developable surface y, then, as is well known,

$$y_i = x_i - a\left(\frac{R}{T}\alpha_i - \gamma_i\right) \qquad (i = 1, 2, 3),$$

where α and γ are the directions of the tangent and binormal at the general point of the curve x, R and T are the radii of curvature and torsion at this point, and a is a parameter. Dr.

Graustein finds as the finite equation of the geodesics on y

$$\left(c_2 + \frac{R}{T}c_3\right)a - (c_1 + c_3 s) = 0,$$

where c_1, c_2, c_3 are arbitrary constants; and as the finite equation of the geodesic circles on y

$$\left(\frac{a}{r} - k_1\right)^2 + \left(\frac{1}{r}\left(s - a\frac{R}{T}\right) - k_2\right)^2 = 1,$$

where r is the radius of geodesic curvature of the general circle and k_1, k_2 are arbitrary constants. These equations afford simple deductions of facts already known and lead to a number of results which seem to be new.

26. Dr. Barrow considers the subject of oriented circles in three dimensions. He makes use of the Plücker coordinates of a circle which are formed from the pentaspherical coordinates of any two spheres intersecting in the circle; and accomplishes the orientation by means of a redundant coordinate defined by a quadratic relation as a two-valued function of the Plücker coordinates. A geometric interpretation for a certain invariant function of the coordinates of two oriented circles is given; and a few theorems are proved about systems of oriented circles whose coordinates satisfy one or more linear relations.

27. The transformation considered by Professor Maclay consists in replacing the exponents of a polynomial by their positive residues, in particular their least positive residues, relative to an arbitrary modulus, and forming the difference of the two polynomials. The effect of successive applications of the transformation and the relations of the resultant polynomials are studied.

28. Mr. Alexander's paper contains a simple method for transforming any algebraic surface birationally into a surface without singularities lying in a space of sufficiently many dimensions. By a direct generalization, the device may be used to resolve the singularities of any n dimensional manifold.

Let $f(x_0, x_1, x_2, x_3) = 0$ be an algebraic surface. The transformation

$$x_i = \frac{\partial f}{\partial x_i} \qquad (i = 0, 1, 2, 3) \tag{1}$$

transforms f into its dual $F(x_0, x_1, x_2, x_3) = 0$. The transformation

$$\xi_{ij} = x_i x_j, \tag{2}$$

together with (1), transforms f into a surface ϕ in 15-dimensional space. It will be found that every singular point p of f which is not a uniplanar point is transformed into a curve l of ϕ and that the multiplicity of each point of l is lower than that of p. A uniplanar point may be simplified by a transformation (2′) analogous to (2). By means of a finite number of transformations (2) or (2′) followed in each case by a projection of the surface back into 3-space, all the singularities may be resolved.

29. Letting $u_0 + u_1 + \cdots + u_n + \cdots$ be any series, de la Vallée-Poussin (*Bulletin de l'Académie Belgique*, 1908) introduced the expressions

$$V_n = \sum_{v=0}^{n} \frac{n!n!}{(n-v)!(n+v)!} u_v,$$

and the original series is summable when $\lim_{n=\infty} V_n$ exists. In the present paper, Professor Gronwall forms the generating identity of these sums, viz., writing $\xi = 4x/(1+x)^2$,

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1-x}{1+x} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n! n!} V_n \xi^n.$$

From this identity, a generalization of Abel's theorem for power series is obtained; it is shown that any series summable by Cesàro's method is also summable by the present method with the same sum; and various other properties of the sums V_n are proved. The theory may be generalized to the sums obtained from the fundamental identity by making

$$\xi = 1 - \left(\frac{1-x}{1+x}\right)^{2m}.$$

30. Professor Gronwall gives a direct proof of the existence and uniqueness of a bounded solution of the integral equation

$$\varphi(x, y) = f(x, y) + \int_0^x A(x, y, s)\varphi(s, y)ds$$
$$+ \int_0^y B(x, y, t)\varphi(x, t)dt + \int_0^x ds \int_0^y C(x, y, s, t)\varphi(s, t)dt,$$

when f, A, B and C are bounded and have their discontinuities regularly distributed in the rectangle $0 \leqq x \leqq a$, $0 \leqq y \leqq b$. This proof avoids the usual decomposition of the rectangle (a, b) into smaller rectangles. An application to partial differential equations of the hyperbolic type is given.

31. If the power series $f(x) = x + a_2x^2 + \cdots + a_nx^n + \cdots$ is such that $z = f(x)$ defines the conformal representation of the unit circle in the x-plane on a simple region in the z-plane, then, as Koebe has shown, there exist upper and lower boundaries for $|f(x)|$ and $|f'(x)|$ when $|x| = r < 1$, these boundaries being independent of the coefficients of $f(x)$.

Professor Gronwall obtains the following approximations to these boundaries:

$$\frac{2}{9}r < |f(x)| < \frac{3r}{(1-r)^2}, \quad \frac{1}{5}\cdot\frac{1-r}{(1+r)^3} < |f'(x)| < 5\cdot\frac{1+r}{(1-r)^3}.$$

These approximations are of the correct order of magnitude, as is shown by the example $f(x) = x/(1-x)^2$.

F. N. Cole,
Secretary.

THE WINTER MEETING OF THE SOCIETY AT CHICAGO.

The thirty-fourth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Monday and Tuesday, December 28–29, 1914, it being the third regular meeting of the Society at Chicago. Eighty persons were in attendance upon the sessions, including the following sixty-four members of the Society:

Professor R. P. Baker, Professor W. H. Bates, Mr. William Betz, Professor G. A. Bliss, Professor Henry Blumberg, Professor Daniel Buchanan, Professor H. E. Buchanan, Dr. Josephine Burns, Professor W. H. Bussey, Professor W. D. Cairns, Professor R. D. Carmichael, Professor A. F. Carpenter, Dr. E. H. Clarke, Professor H. E. Cobb, Professor D. R. Curtiss, Dr. W. W. Denton, Professor L. E. Dickson, Mr. C. R. Dines, Professor L. W. Dowling, Professor Arnold Dresden, Professor Arnold Emch, Professor A. B. Frizell, Dr. M. G. Gaba, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Mr. L. A. Hopkins, Professor A. M. Kenyon, Dr. H. R. Kingston, Professor Malcolm McNeill, Professor W. D. MacMillan, Professor H. W. March, Professor J. L. Markley, Dr. T. E. Mason, Professor G. A. Miller, Professor U. G. Mitchell, Professor E. H. Moore, Professor E. J. Moulton, Professor L. C. Plant, Mr. V. C. Poor, Professor S. W. Reaves, Professor W. J. Risley, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Dr. H. M. Sheffer, Mr. T. M. Simpson, Professor C. H. Sisam, Professor E. B. Skinner, Professor H. E. Slaught, Professor C. S. Slichter, Professor R. B. Stone, Professor E. B. Stouffer, Professor A. L. Underhill, Professor E. B. Van Vleck, Dr. G. E. Wahlin, Professor Mary E. Wells, Professor Marion B. White, Professor E. J. Wilczynski, Professor K. P. Williams, Professor R. E. Wilson, Mr. C. H. Yeaton, Professor J. W. A. Young, Professor W. A. Zehring, Professor Alexander Ziwet.

Professor E. B. Van Vleck, President of the Society, presided at the session on Monday afternoon. Professor E. J. Wilczynski, chairman of the Section and Vice-President of the Society, presided at all the other sessions, except when relieved by Professor C. S. Slichter during part of the session on Monday morning.

At the business meeting of the Chicago Section the report of the committee of the Section appointed last April to consider the relation of the Society to the field now covered by the *American Mathematical Monthly* was received and ordered spread upon the minutes. In accordance with the recommendation of this committee, it was then voted that the Chicago Section request the Council of the Society to appoint a committee to consider and report concerning possible relations of the Society to the field now covered by the *American Mathematical Monthly.* In transmitting this request to the Council the Secretary of the Section was instructed to send with it a copy of the report of the committee of the Section.

The following officers of the Chicago Section were elected: Chairman, Professor E. J. Wilczynski; Secretary, Professor H. E. Slaught; and third member of the program committee, Professor R. D. Carmichael. Professor Wilczynski was requested by the Section to deliver an address upon the completion of his second term of office as chairman.

The usual social gathering and dinner on Monday evening was attended by sixty members, this being an unusually large number.

The following papers were presented at this meeting:

(1) Professor **Arnold Emch**: "On some general theorems concerning analytic curves."

(2) Professor A. F. **Carpenter**: "Ruled surfaces with plane flecnode curves."

(3) Professor E. J. **Wilczynski**: "The general theory of congruences."

(4) Professor A. B. **Frizell**: "A non-enumerable well-ordered set of infinite permutations."

(5) Professor **Henry Blumberg**: "On a generalization of certain theorems relating to discontinuous functions."

(6) Professor **Daniel Buchanan**: "A new isosceles triangle solution of the three-body problem."

(7) Professor **Daniel Buchanan**: "Oscillations near an isosceles triangle solution as the finite bodies become unequal."

(8) Mr. L. L. **Steimley**: "On the general solution and so-called special solutions of linear homogeneous partial differential equations."

(9) Mr. W. L. **Hart**: "A method of successive approximations for the determination of implicit functions."

(10) Professor **Arnold Dresden**: "On the second derivative of the extremal integral for a general class of problems."

(11) Professor G. A. **Bliss**: "A note on functions of lines."

(12) Professors G. A. **Bliss** and A. L. **Underhill**: "Shortest lines between curves and surfaces."

(13) Dr. G. R. **Clements**: "Singular point transformations in two complex variables. Supplementary note."

(14) Professor **Henry Blumberg**: "On the factorization of certain polynomials and certain linear homogeneous differential expressions. Second communication."

(15) Dr. G. R. **Clements**: "The resultant of a system of polynomials."

(16) Mr. A. R. **Schweitzer**: "On the formal properties of functional equations."

(17) Professor L. E. **Dickson**: "On modular cubic surfaces."

(18) Professor G. A. **Miller**: "A new proof of Sylow's theorem."

(19) Professor G. A. **Miller**: "Note on the potential and the antipotential group of a given group."

(20) Professor J. B. **Shaw**: "On parastrophic algebras."

(21) Mr. C. H. FORSYTH: "A general formula for the valuation of bonds."

(22) Dr. H. R. KINGSTON: "Metric differential properties of nets of plane curves."

Mr. Steimley was introduced by Professor R. D. Carmichael, Mr. Hart by Professor F. R. Moulton, and Mr. Forsyth by Professor J. W. Glover. The papers of Dr. Clements and Mr. Schweitzer were read by title.

Abstracts of the papers follow, the numbers corresponding to those of the titles above.

1. In an article which will appear in the *Rendiconti del Circolo Matematico di Palermo* it is proved analytically that every closed analytic curve without cusps and multiple points has an even number of points of inflexion and admits of an even number of tangents parallel to any given direction. The proofs of these theorems are based upon the co-periodicity of the parametric representation of such curves and of their derivatives.

These results may be extended to any closed analytic curve by properly extending the definition of inflexions and tangents and by taking proper care of multiplicities. Among such closed curves may be included those that result from the former by projective transformations.

A purely intuitive proof of the theorem on inflexions was given by von Staudt in his Geometrie der Lage. In the present paper Professor Emch extends his former proofs to the general cases and discusses von Staudt's method.

2. In the general theory of ruled surfaces as developed by Wilczynski, many important theorems are obtained by taking for directrix curves the two branches of the flecnode curve. In Professor Carpenter's paper the condition that these curves be plane leads to the theorem: If the two branches of the flecnode curve of a ruled surface are distinct plane curves, then the two branches of the complex curve are plane and the four planes of these curves form a harmonic pencil of planes in which the planes of the two branches of the flecnode curve are separated by the planes of the two branches of the complex curve.

3. Professor Wilczynski's paper is concerned with the projective differential properties of congruences of rays.

By introducing a number of new congruences covariantly related to the given one, many new theorems present themselves and most of the familiar ones appear in a new light. Incidentally a new geometric interpretation is found for the condition that the Laplace-Darboux invariants of a conjugate system be equal. There is also found the geometrical significance of the systems which Bianchi calls isothermally conjugate.

4. In a previous paper (Chicago, April, 1914) Professor Frizell has called attention to a non-denumerable well-ordered set of finite permutations. The present note applies this result to obtain a non-enumerable well-ordered set of infinite permutations of the series of natural numbers. This is done by arranging the natural numbers in series higher than ω.

5. In his first paper, Professor Blumberg shows that almost all of the known general theorems on real, discontinuous functions involving the oscillation function (which is associated with every given function, whether discontinuous or continuous) hold for the case where the range of the independent variable is an abstract aggregate $\mathfrak{S}$ subject to two simple conditions. The first demands for every element s of $\mathfrak{S}$ the existence of a sub-aggregate $\mathfrak{M}$ of $\mathfrak{S}$ having the relation R to s, R being an undefined relation. The second says that if $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are two "neighborhoods" (the term "neighborhood" having been defined in terms of the undefined relation R) of an element s of $\mathfrak{S}$, there exists a common sub-aggregate $\mathfrak{N}$ of $\mathfrak{N}_1$ and of $\mathfrak{N}_2$ that is a "neighborhood" of s. In particular, for example, the Sierpinski theorem holds. Certain general theorems on discontinuous functions communicated by the author in former papers are similarly generalized.

6. The isosceles-triangle solutions of the three body problem are the periodic solutions in which two of the bodies have equal mass, and the third body moves so that its distances from the other two bodies are always equal. The third body, then, moves in a straight line which passes through the center of gravity of the system and is perpendicular to the plane of the initial motion of the equal masses. In a paper presented to the Society at Chicago, April, 1911, Professor Buchanan discussed the periodic solutions in which the two

equal masses move in ellipses and the third body is infinitesimal. Two periodic solutions were shown to exist. They are expansible as power series in a certain parameter λ, which is introduced through the factor representing the mass of the finite bodies, and the coefficients are power series in e, the eccentricity of the ellipses. The period T is a function of λ but not of e. In the present paper two periodic solutions, having period T, are shown to exist when the third body becomes finite. They are expansible as power series in a parameter representing the mass of the third body, and the coefficients are power series in λ and e similar to those appearing in the previous paper.

7. Starting with the simplest case of the isosceles-triangle solutions in which the third body is infinitesimal and the finite bodies move in circles, Professor Buchanan in his second paper deals with the oscillations near this solution as the finite masses become unequal. If the finite bodies have masses $m + \mu$ and $m - \mu$, where the unit of mass is chosen so that $m = \frac{1}{2}$, then the solutions are expansible as power series in μ. The coefficients are power series in another parameter which represents the initial projection of the infinitesimal body from the plane of motion of the finite bodies where $\mu = 0$. Two solutions are shown to exist and they are expansible as power series in $\mu^{\frac{1}{2}}$ or μ according as the purely imaginary characteristic exponents arising from the equations of variation are imaginary integers or not.

8. It is known that the customary classification of the integrals of linear partial differential equations is incomplete even in the case where the coefficients are single valued and analytic in a given domain. For this latter case Mr. Steimley gives a complete classification of all possible integrals and exhibits a solution which is general in the sense that it includes all possible solutions (even the so-called special solutions).

9. Let there be given the system of equations

$$(1) \qquad f_i(x_j; \mu) = 0 \qquad (i, j = 1, 2, \cdots, n),$$

in the field of reals, with the solution point $(a_j; \mu_0)$ at which the Jacobian, $\partial(f_1, \cdots, f_n)/\partial(x_1, \cdots, x_n)$, is different from zero. The current methods by which the fundamental theorem of implicit function theory is proved for system (1), all possess the fault that no practical scheme is given for the computation of the values of the $x_j(\mu)$ except for μ sufficiently near to μ_0. In Mr. Hart's paper there is given a method of successive approximations which furnishes a practical scheme for computing the values of the $x_j(\mu)$ over a range in general much more extended than that obtained in the usual proofs.

10. In a paper published in volume 9 of the *Transactions* of the Society, Professor Dresden treated problems of the calculus of variations by reducing them to problems in ordinary maxima and minima by means of the "extremal integral." This necessitated securing formulas for the second derivatives of this integral; these were obtained in the paper mentioned above and they are quoted on page 312 of Bolza's Vorlesungen. It is the purpose of the present paper to secure similar formulas for more general problems of the calculus of variations, viz., such as involve integrals whose integrand contains n unknown functions and their first derivatives, in non-parametric form as well as in parametric form. In the former case these formulas are obtained without great difficulty. In the latter case, however, it has seemed to the author that a reduction of the second variation was necessary. This is done by means of a reduction of differential forms discussed in a paper presented to the Society some time ago and published in the *Annals of Mathematics* for March, 1912.

It is intended to use these formulas for the treatment of the maxima and minima of integrals of the type referred to above when one or both end points are variable, as well as when broken extremals are admitted. Furthermore the cases are to be taken up in which the unknown functions are also to satisfy a system of differential equations.

11. One of the most important applications of the theory of functions of lines is in the calculus of variations, where the integrals considered are all either functions of this sort or else quite natural generalizations of them. Unfortunately the usual definitions of continuity and the derivative of a function of a line do not apply to such integrals. In the paper of Professor Bliss the character of the continuity of an integral of the calculus of variations is discussed and the inadequacy of the definition of the derivative is shown.

Attention is called to the modifications which make these notions applicable at once to such line functions.

12. If an arc E in space joins two surfaces S and S' and is shorter than any other arc in its neighborhood joining S with S', then E must in the first place be a straight line, and in the second place it must cut the two surfaces orthogonally. These conditions are necessary but not sufficient to insure the minimizing property. The further relations which must be satisfied involve the curvatures of the surfaces at their intersections with the line E. In the paper of Professors Underhill and Bliss these relations are deduced, and analogous ones are found for the cases where the end points of E lie upon two curves, or on a surface and a curve. The results are of interest because of the preliminary information which they give with respect to the more general problem of determining the minimizing arcs, with end points variable on curves or surfaces, for an integral of the form

$$\int f\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right) dx.$$

13. In the *Annals of Mathematics* for September, 1913, Dr. Clements discussed the transformation

$$T: \qquad x = f(u, v), \qquad y = \varphi(u, v)$$

in the following cases:

(*a*) $f(u, v)$ and $\varphi(u, v)$ denote functions of the complex variables u and v, single-valued and analytic throughout a neighborhood R of $u = 0$, $v = 0$;

$$(b) \qquad f(0, 0) = 0, \qquad \varphi(0, 0) = 0;$$

$$(c) \qquad J(u, v) \equiv f_u\varphi_v - f_v\varphi_u \not\equiv 0, \qquad J(0, 0) = 0.$$

In that paper the discussion was not complete for the case in which $f(u, v)$ and $\varphi(u, v)$ have a common factor in the point $(0, 0)$ but neither is a factor of the other in this point. In the present note it is shown that for this case the inverse of T exists and is finitely multiple-valued and continuous throughout a complete neighborhood of the point $x = 0$, $y = 0$, with the exception of that point itself, which explodes into a locus in the uv-region. This establishes the existence of functions of two complex variables, finitely multiple valued and continuous except in isolated points.

Any transformation T can always be replaced either by a transformation of the form $x = u$, $y = v^n$ (n a positive integer), a finite number of the form $x = u$, $y = uv$, and of those one-to-one and analytic both ways; or by the preceding combined with a transformation T for which $f(u, v)$ and $\varphi(u, v)$ have no common factor.

14. In his second paper, Professor Blumberg gives criteria for the resolvability and, in particular, the irreducibility of certain polynomials, linear homogeneous differential expressions, and other expressions. The methods used are the simplest heretofore employed in similar investigations; the results are the most general heretofore obtained and, largely on account of the simplicity of the methods, generalization is made easy. The most general theorem communicated by the author at the November, 1914, meeting of the Southwestern Section is a special case of a special case of one of the theorems in the present paper. To indicate the character of the results one of the theorems obtained and several special cases of it are given below. Let

$$P = a_0(x)\frac{d^p y}{dx^p} + a_1(x)\frac{d^{p-1} y}{dx^{p-1}} + \cdots + a_p(x) y$$

be a linear homogeneous differential expression belonging to (i. e., whose coefficients belong to) the domain $\mathfrak{R}$ of rational functions; let d_ν ($\nu = 0, 1, \cdots, p$) represent the degree of a_ν; and let P be decomposable into the symbolic product RS, where R and S are linear homogeneous differential expressions belonging to $\mathfrak{R}$ and of the rth and sth orders respectively. The following theorem holds: If for a fixed κ (> 0)

$$\frac{d_\kappa - d_0}{\kappa} \geqq \frac{d_\nu - d_0}{\nu} \qquad (\nu = 1, 2, \cdots, p)$$

and

$$d_\kappa - d_0 \geqq 0,$$

at least one of the following $n - \kappa + 1$ congruences holds:

$$d_\kappa - d_0 \equiv 0 \left(\mathrm{mod}\ \frac{\kappa}{[\kappa, \rho]}\right),$$

where ρ takes the successive values $n - r, n - r + 1, \cdots, \kappa - r$ (or $n - s, n - s + 1, \cdots, k - s$), and $[\kappa, \rho] =$ G.C.D. of κ and ρ. A special case of this theorem is the following: If

$$\frac{d_\kappa - d_0}{\kappa} \geqq \frac{d_\nu - d_0}{\nu} \qquad (\nu = 1, 2, \cdots, p),$$

$d_\kappa - d_0 \geqq 0$ and $[d_\kappa - d_0, \kappa] = 1$ (i. e., $d_\kappa - d_0$ and κ are relatively prime), either $r \geqq \kappa$ or $s \geqq \kappa$. From this special case follows the theorem: If

$$\frac{d_p - d_0}{p} \geqq \frac{d_\nu - d_0}{\nu} \qquad (\nu = 1, 2, \cdots, p),$$

$d_p - d_0 \geqq 0$ and $[d_p - d_0, p] = 1$, P is irreducible.

The above theorems and nearly all others obtained hold for polynomials whose coefficients are rational functions of one or more variables, for linear difference equations, and also for certain cases where the domain of rationality is more general than $\mathfrak{R}$, the only changes required in the formulation of the theorems being the substitution, for example, of "difference" for "differential" in passing to "difference equations," etc.

One of the theorems generalizes the most general criterion for the irreducibility of polynomials heretofore known, a criterion due to Perron.

15. It is well known that in any *numerical* case there exist $N - 1$ rational integral functions of the coefficients of a system of polynomials

$$f_n(x) \equiv A_{n0}x^{m_n} + \cdots + A_{nm_n} \qquad (n = 1, \cdots, N)$$

whose simultaneous vanishing is the necessary and sufficient condition that these N polynomials shall have a common factor of the first or higher degree in x. Further, if $N = 2$, this resultant can be written formally in terms of the coefficients. Dr. Clements points out that when $N > 2$, if the coefficients are real, there exist $N - 1$ rational integral functions, expressible as formulas in the coefficients and hence in their definition independent of the numerical value of these coefficients, whose simultaneous vanishing is the necessary and sufficient condition that $f_1, \cdots, f_N$ shall have a common factor containing x. Further, infinitely many such sets of resultants can be found. A very simple set of resultants is exhibited for the case $N = 3$, $m_3 = 2$.

16. In the first part of Mr. Schweitzer's paper a set of postulates is given for a *field* in which the explicit assumption of existential properties is avoided. These postulates are stated in terms of four indefinables, $f(x, y)$, $\phi(x, y)$, $\phi_1(x, y)$, $f_1(x, y)$, and are obtained essentially by giving abstract interpretations to functional relations discussed in a previous paper.*

In the second part of the paper the following theorems are proved:

I. If $f\{\beta f(x, z), f(x, y)\} = \alpha\beta f(y, z)$, then there exists $\chi(x)$ such that $\chi f(x, y) = c_0\chi(x) - c_0\chi(y)$, $\chi\alpha(x) = c_0\chi(x)$, and $\chi\beta(x) = \chi(x) + c_1$, where c_0 and c_1 are constants.

II. If

$$f\{f(x, z), f(y, z)\} = \alpha f(x, y)$$

and

$$f_1\{\alpha f_1(x, z), f_1(y, z)\} = \alpha f_1(x, y),$$

then there exists $\chi(x)$ such that $\chi f(x, y) = c_0\chi(x) - c_0\chi(y)$, $\chi\alpha(x) = c_0\chi(x)$ and $\chi f_1(x, y) = c_1\chi(x)/\chi(y)$, that is, $\alpha^{-1}f(x, y)$ and $f_1(x, y)$ are quasi-transitive.

III. If $f\{f_1(x, z), f_1(y, z)\} = f_1(x, y)$ and $f_1\{f(x, z), f(y, z)\} = f(x, y)$, then $f(x, y)$ and $f_1(x, y)$ are identical.

17. The method employed in the paper by Professor Dickson for the investigation of surfaces of any order in modular space is based upon the projective classification of all sets of real points (i. e., points with integral coordinates). For the case of modulus 2, there are exactly $n - 1$ projectively distinct types of sets of n real points when $1 < n < 8$. Since there are just 15 real points, the number of types of sets of n points is the same as the number of types of sets of $15 - n$ points. Hence a case $n \geqq 8$ reduces to a case $n < 8$. To investigate the projectively distinct types of surfaces of a given order with integral coefficients taken modulo 2, we treat as a separate case those surfaces the only real points on which are those of a given canonical set of n real points ($n = 0, 1, \cdots, 15$), so that we have 15 linear relations between the coefficients of the

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* Cf. BULLETIN, October, 1914, p. 25, (6); p. 26, (11); p. 28, the relation following (22).

equation of the surface. The transformations available for the further specialization of the coefficients are the automorphs of the set of n points.

For a cubic surface modulo 2, it is shown that the number of real points n on the surface is always odd. For $n = 15$, the surface is equivalent to $xy(x + y) = 0$ or to $xy(x + y) = zw(z + w)$, the latter having no singular point and having 27 straight lines, of which 15 are real. For $n = 1$, there is a single type of surface; it has no singular point and no real line. For $n = 13$, one of the two types has a singular point, while the other is equivalent to

$$xy(x + y) = w(w + z)(y + z),$$

9 of whose 27 lines are real. On one of the five types for $n = 3$, just one of the 27 lines is real. For $n = 11$, there is a type 5 of whose 27 lines are real. Thus there are types of cubic surfaces modulo 2 having exactly 0, 1, 5, 9 or 15 real lines. In the algebraic theory, the only possible numbers for non-singular cubic surfaces are 3, 7, 15, 27. The paper will appear in the *Annals of Mathematics*.

18. Professor Miller bases his new proof of Sylow's theorem on the following theorem: The number of the substitutions of degree p^β and of order p in the symmetric group of degree n is prime to p whenever p^β is the highest power of p which does not exceed n. By means of this elementary theorem Sylow's theorem may be proved as follows: Suppose that the order g of any group G is divisible by p^α but not by $p^{\alpha+1}$. Now represent G as a regular substitution group, and construct all the substitutions on the g letters of G such that these substitutions are of order p and of degree p^β, p^β being the highest power of p which is less than n.

These substitutions are transformed under G into complete sets of conjugates, each set involving more than one. At least one of these sets must involve a number m of substitutions such that m is prime to p, since the total number of these substitutions is prime to p. Hence G contains a subgroup whose order is divisible by p^α. If the order of this subgroup exceeds p^α the given operations may be repeated, and hence we can always reach a subgroup of order p^α, by the given process, provided the order of G is divisible by p^α.

19. Professor Miller's second paper was published in full in the February BULLETIN.

20. The parastrophic algebra is defined with regard to a given algebra by the relation between the multiplication constants

$$\gamma'_{ijk} = \gamma_{ikj} \qquad (i, j, k = 1, \cdots, r),$$

where r is the order of the algebra. The parastrophic algebra is usually not associative when the original algebra is associative. In the case of algebras defined by finite groups the parastrophic algebra takes a very simple form. Professor Shaw's paper discusses the relations between these algebras.

21. In Mr. Forsyth's paper a general formula is derived for the valuation of bonds where the principal of S dollars is to be repaid in $r_1, r_2, \cdots, r_n$ installments, of which the first r_1, the next r_2, etc., are equal, there being n *different* installments. All other similar formulas apply only to cases involving equal installments.

The first installment is to be paid at the end of f_1 years, the next $r_1 - 1$ at intervals of t_1 years, etc., and in general if we designate as "major intervals" those intervals wherein all the installments are equal, there will be r_n installments at intervals of t_n years in the nth major interval, the first installment to be paid at the end of f_n years.

The rate of interest offered is g and the rate to be realized i, both to be paid m times a year.

The premium or discount becomes

$$(1) \qquad K = \left[1 - \frac{\sum_1^n a : n}{s}\right]\left(\frac{g - i}{i}\right),$$

where

$$a : n = A_n\{a_m(f_n + r_nt_n) - a_mf_n\}, \qquad A_n = I_n/a_{mt_n}$$

and I_n represents the value of each of the installments in the nth major interval. The a's with subscripts refer to the ordinary annuity certain.

If all the minor intervals are equal ($= t$), formula (1) reduces to

$$(2) \qquad K = \left[1 - \frac{\sum_0^n (I_n - I_{n+1})a(n)}{sa_{mt}}\right]\left(\frac{g - i}{i}\right),$$

where

$$a(n) = a_m\left(f_1 + t\sum_1^n r_n\right)$$

and

$$I_0 = I_{n+1} = 0.$$

22. In a memoir entitled "One-parameter families of nets of plane curves" Professor Wilczynski has studied the projective differential properties of nets of plane curves, by means of a completely integrable system of three linear partial differential equations of the second order. In Dr. Kingston's paper some of the metric differential properties of these nets are discussed. If the two linearly independent, non-constant solutions of the partial differential equations

$$y_{uu} = ay_u + by_v, \quad y_{uv} = a'y_u + b'y_v, \quad y_{vv} = a''y_u + b''y_v$$

be taken as the cartesian coordinates of the moving point, it is shown that for a given set of coefficients $a, b, \cdots, b''$, which are functions of the independent variables u and v and are subject to certain integrability conditions, these equations determine a net uniquely except for an affine transformation. If, moreover, we consider the fundamental quantities E, F, and G, which are connected with the length of arc by the relation $ds^2 = Edu^2 + 2Fdudv + Gdv^2$, and satisfy certain conditions, and if we add the condition that these shall have given values, then our original equations along with the given E, F, and G, determine a net uniquely except for a motion and a reflection. The application to orthogonal and isothermal nets gives rise to interesting results. In this paper is laid also the foundation for a detailed study of the metric properties of those nets which are composed of two families of conics.

H. E. SLAUGHT,
Secretary of the Chicago Section.

THE STRUCTURE OF THE ÆTHER.

BY DR. H. BATEMAN.

(Read before the American Mathematical Society, February 28, 1914.)

§ 1. *Introduction.*

IN Maxwell's electromagnetic theory of light the broad view is taken that waves of light are manifestations of certain electromagnetic phenomena in the æther which are included in the general class of processes capable of being described by means of the partial differential equations

$$(1) \qquad \text{rot } M = -\frac{i}{c}\frac{\partial M}{\partial t}, \quad \text{div } M = 0.$$

In these equations M denotes the complex vector $H + iE$, the vectors E and H represent the electric and magnetic intensities respectively, c is a constant which is usually called the velocity of light, and rot M denotes the complex vector whose components are of type

$$\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}$$

when right-handed rectangular axes are used.

In the extensions of the theory which have been made by H. A. Lorentz* and Sir Joseph Larmor,† it is recognized that the fundamental equations (1) must not be regarded as holding for all real values of the variables x, y, z, t; the exceptional domains or singular points are, moreover, regarded as instrumental in the production of waves or disturbances in the æther.

The hypothesis generally adopted by Lorentz is that there are exceptional domains, which, when viewed at any instant, consist of an enormous aggregate of small closed regions. These domains are supposed to be occupied by electricity whose volume density ρ is generally a function of x, y, z, t. The motion of the electricity is supposed to be specified by a vector function v.

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* *Archives néerlandaises*, vol. 25 (1892); *Amsterdam Proceedings* (1902), p. 305.

† *Phil. Trans.* A, vol. 185 (1894), p. 719; vol. 186 (1895), p. 695; Æther and Matter, Cambridge (1900).

The vector functions E and H are defined for all real values of $x, y\ z, t$ with the aid of the retarded potentials of L. Lorenz. These potentials A and Φ are expressed as triple integrals involving ρ and v and the vectors E, H are derived from them with the aid of the relations

$$(2) \qquad E = -\frac{1}{c}\frac{\partial A}{\partial t} - \frac{\partial \Phi}{\partial x}, \qquad H = \text{rot } A.$$

When E and H are defined in this way they are found to satisfy equations (1) when the space-time point (x, y, z, t) does not belong to the exceptional domains: for points in the regions occupied by electricity, the modified equations

$$(3) \qquad \text{rot } M = \rho v - \frac{i}{c}\frac{\partial M}{\partial t}, \qquad \text{div } M = i\rho$$

are satisfied.

Starting from these equations Lorentz and Larmor have obtained, by a process of averaging, a set of equations capable of describing the principal optical and electromagnetic properties of a material medium. These more complicated equations are usually called the macroscopic equations, and equations (3) the microscopic equations or fundamental equations of the theory of electrons.

Sir Joseph Larmor has used a more fundamental theory* in which the simple equations (1) are assumed to be the whole scheme and electricity is regarded as consisting of so-called isolated point singularities in the æther, that is, points at which E and H become infinite in a certain manner. A solution of equations (1) which can be regarded as giving an exact specification of an elementary electromagnetic field with a single isolated point singularity of the required type has been obtained by A. Liénard† for the case in which the velocity of the singularity is always less than that of light. It turns out that the electric charge associated with the singularity does not vary with the time. With the aid of this solution, which was originally derived from the retarded potentials for a volume distribution of electricity, it is possible to reverse matters and effect a transition from equations (1) to equations (3) by a process of averaging or superposition. The exceptional domains within which ρ and ρv are different from zero are now supposed to be occupied by a closely packed aggregate of the point singularities of the superposed elementary electromagnetic fields.

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* Æther and Matter, p. 77.

† *L'Eclairage électrique*, t. 16 (1898), pp. 5, 53, 106.

One advantage of this more fundamental theory is that it emphasizes two properties of electricity, namely the invariance of the electric charge and the limitation to which its velocity of motion is subject.* The mathematical discussion of the theory is at present incomplete, partly because some hypothesis is needed which will reveal the structure of the æther and show clearly that electric charges are indeed point singularities in the medium. In this note an hypothesis is considered which fulfils some of the requirements; the mathematical analysis is, however, still incomplete.

§ 2. *Vector Fields with Moving Singular Curves.*

I have shown elsewhere† that there are vector fields satisfying equations (1) in which the vector functions E, H become infinite at points lying on certain moving curves. There is, moreover, at present no evidence that these fields can be built up by superposition from elementary electromagnetic fields of Liénard's type. I propose, therefore, to call them *æthereal fields* so as to distinguish them from the *electromagnetic fields* which possess the property just mentioned. Whether æthereal fields have a physical existence or not is an open question, but I propose to consider the hypothesis that when a large number of æthereal fields are superposed their singular curves indicate the structure of an "æther" which is capable of supporting a certain type of electromagnetic field which is mathematically determined by the properties of the superposed æthereal fields.

According to this view the structure of the æther depends

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*This limitation was first clearly pointed out by J. J. Thomson, Recent Researches (1893), p. 21. See also O. Heaviside, *Phil. Mag.*, vol. 27 (1889), p. 324.

† *Philosophical Magazine*, Oct., 1913, Jan., 1914; *Annals of Mathematics*, 1914; The Mathematical Analysis of Electrical and Optical Wave Motion on the Basis of Maxwell's Equations, Cambr. Univ. Press, 1914, Ch. 8. This will be denoted hereafter by E.

to some extent upon the type of electromagnetic field which it supports, and vice versa. There must consequently be some definite mathematical relation between the electromagnetic field and the superposed æthereal fields whose singular curves indicate the structure of the æther. We shall now consider a relation which is suitable for the case in which the electromagnetic field is an elementary field of Liénard's type.

§ 3. *Conjugate Fields.*

Two vector fields in which the complex vectors are M, M' respectively are said to be conjugate when the scalar product (MM') is zero. A field which is conjugate to itself is said to be self-conjugate and in this case $(M^2) = 0$.

If the direction of the flow of energy in two fields is the same at each point and both fields are self-conjugate they are also conjugate to one another. To prove this we remark that since the direction of the flow of energy is at right angles to both the electric and magnetic intensities, the vectors E, H, E', H' associated with any point must lie in one plane. Moreover, since both fields are self-conjugate the electric and magnetic intensities in each field are perpendicular, hence we must have $M' = kM$, where k is a complex quantity. It is now clear that the relation $(MM') = 0$ is a consequence of the two relations $(M^2) = 0$, $(M'^2) = 0$. It should be noticed that the present argument fails if the flow of energy takes place in *opposite* directions in the two fields, because then the correct conclusion is that $M' = kM^*$ where

$$M^* = H - iE.$$

Let us now consider an electromagnetic field and a large number of mutually conjugate self-conjugate æthereal fields each of which is conjugate to the electromagnetic field. We shall regard the singular curves of these æthereal fields as forming an *element of the æther* and shall suppose that it is this element which supports the particular electromagnetic field in question. In the example of an element which will be given presently the singular curves meet in a point which is the point singularity of the electromagnetic field supported by the element.

It should be remarked that the condition of conjugacy is invariant when a transformation of variables is made which leaves equations (1) unaltered in form. Our hypothesis is thus compatible with the modern ideas of relativity. It should also be noticed that when a number of mutually conjugate fields are superposed the term of type $(E^2) - (H^2)$ in the Lagrangian function of the total field is the sum of the terms of this type belonging to the different constituent fields. It is possible then that the condition of conjugacy has a dynamical significance. The fields used in forming an element of the æther can be regarded as analogous to the terms of a Fourier series or the elements of a Fourier integral; each constituent field is supposed to be multiplied by a factor or coefficient, and by giving different values to these coefficients different states of vibration of the fibers of the element may be represented mathematically. With regard to the nature of this vibration or disturbance which can run along a singular curve or fiber the only certain information we are able to give at present is that its orientation may vary with the time and be different for different fibers.

It is by no means certain that the fibers of an element should be regarded as filling the whole of space. If we consider only those æthereal fields whose singular curves lie within a kind of tube or cone having its vertex at the point singularity of the electromagnetic field, it appears likely, judging from some simple cases which have been considered,* that it is possible to choose the coefficients so that when both the electromagnetic and æthereal fields are superposed the vectors E, H in the total field are zero outside the tube. This corresponds to the type of element which has been considered by Sir Joseph Thomson in his theory of the structure of the electric field.†

§ 4. *A Solution of the Fundamental Equations.*

The components of a vector M which satisfies the equations (1) are known to be solutions of the wave equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{4}$$

The type of singular curve which is typical of an æthereal field is obtained by considering the case in which the components of M are all wave functions of type $u = \gamma f(\beta)$, where the function f is different for each component but the functions β and γ are the same. The quantities β and γ are supposed to be functions of x, y, z, t; an essential property of β is that it must be complex so that the equation $1/f = 0$ may be equivalent to two real relations connecting x, y, z, t and so generally represent a moving curve. It has been shown that the velocity of this singular curve in a direction at right angles to the tangent at any point is never greater than the velocity of light.*

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* See for instance E, p. 121.

† Recent Researches, Ch. I; *Phil. Mag.*, vol. 19 (1910), p. 301; Ibid., Oct., 1913, Dec., 1913. See also N. R. Campbell, *The New Quarterly*, 1909; D. N. Mallik, *Phil. Mag.*, vol. 26 (1913), p. 144.

The problem of finding all solutions of the wave equation having the general form $\gamma f(\beta)$ has not been solved completely, but in the particular case in which this expression can be regarded as included in a more general expression of type $u = \gamma f(\alpha, \beta)$, where f is an arbitrary function and u satisfies the wave equation, the problem can be solved as follows:

It is easily seen that the function f must satisfy the differential equation

$$(5) \qquad \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \frac{1}{c^2}\left(\frac{\partial f}{\partial t}\right)^2$$

which gives rise to three equations connecting α and β when the coefficients of

$$\left(\frac{\partial f}{\partial \alpha}\right)^2, \quad \frac{\partial f}{\partial \alpha}\frac{\partial f}{\partial \beta}, \quad \text{and} \quad \left(\frac{\partial f}{\partial \beta}\right)^2$$

are equated to zero.

Now let x, y, α, β be taken as new independent variables; the preceding equation then takes the form

$$(6) \qquad K^{-2}[AX^2 - 2HXY + BY^2] = 0,$$

where

$$A = \left(\frac{\partial t}{\partial x}\right)^2 + \left(\frac{\partial t}{\partial y}\right)^2 - \frac{1}{c^2}, \qquad B = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1,$$

$$H = \frac{\partial z}{\partial x}\frac{\partial t}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial t}{\partial y}, \qquad K = \frac{\partial(z, t)}{\partial(\alpha, \beta)},$$

$$X = \frac{\partial(f, z)}{\partial(\alpha, \beta)}, \qquad Y = \frac{\partial(f, t)}{\partial(\alpha, \beta)}.$$

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* E, p. 123.

Equating to zero the coefficients of

$$\left(\frac{\partial f}{\partial \alpha}\right)^2, \quad \frac{\partial f}{\partial \alpha}\frac{\partial f}{\partial \beta}, \quad \left(\frac{\partial f}{\partial \beta}\right)^2,$$

we obtain three linear equations connecting A, B, H. The determinant of the coefficients is easily found to be K^{-3}, and this is zero only when there is a relation between α, β, x, y. Ignoring this trivial case, we may conclude from (6) that $A = 0$, $B = 0$, $H = 0$. Now in virtue of the last equation we may write

$$\frac{\partial t}{\partial x} = \lambda \frac{\partial z}{\partial y}, \quad \frac{\partial t}{\partial y} = -\lambda \frac{\partial z}{\partial x},$$

where λ is some function of x, y, α, β. Substituting in the equation $A = 0$, we find that $c^2\lambda^2 + 1 = 0$, hence

$$c\frac{\partial t}{\partial x} = i\frac{\partial z}{\partial y}, \quad c\frac{\partial t}{\partial y} = -i\frac{\partial z}{\partial x}.$$

These equations imply that

$$z - ct = F(x+iy), \quad z + ct = G(x-iy),$$

where F and G are functions which depend also on α and β. Substituting these expressions in the equation $B = 0$, we find that

$$F'(x+iy)G'(x-iy) + 1 = 0.$$

Hence we may put

$$(7) \quad z - ct = \phi + \theta(x+iy), \quad z + ct = \psi - \frac{1}{\theta}(x-iy),$$

where θ, ϕ, ψ are arbitrary functions of α and β. These equations may be used to define α and β as functions of x, y, z, t. It has already been shown* that when α and β are defined by equations of this type a solution of the wave equation (4) may be obtained by writing $u = \gamma f(\alpha, \beta)$ where γ is one of the Jacobians of type $\partial(\alpha, \beta)/\partial(y, z)$. The preceding analysis indicates that all the wave functions of this type may be obtained by the method just described; to complete the proof we must show that the ratio of two suitable values of γ depends only on α and β.

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* *Messenger of Mathematics*, March, 1914, p. 164; *E*, p. 134.

Let us suppose that $\gamma f(\alpha, \beta)$ and $\gamma' F(\alpha, \beta)$ satisfy the wave equation, where f and F are arbitrary; then it is easy to deduce that

$$\frac{\partial w}{\partial x}\frac{\partial \alpha}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial \alpha}{\partial y}+\frac{\partial w}{\partial z}\frac{\partial \alpha}{\partial z}=\frac{1}{c^2}\frac{\partial w}{\partial t}\frac{\partial \alpha}{\partial t},$$

$$\frac{\partial w}{\partial x}\frac{\partial \beta}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial \beta}{\partial y}+\frac{\partial w}{\partial z}\frac{\partial \beta}{\partial z}=\frac{1}{c^2}\frac{\partial w}{\partial t}\frac{\partial \beta}{\partial t},$$

where $w = \gamma'/\gamma$. These equations imply that

$$\frac{\partial w}{\partial x}=P\frac{\partial \alpha}{\partial x}+Q\frac{\partial \beta}{\partial x},\quad \frac{\partial w}{\partial y}=P\frac{\partial \alpha}{\partial y}+Q\frac{\partial \beta}{\partial y},\quad \text{etc.};$$

for $w = \alpha$ and $w = \beta$ are evidently particular solutions of the preceding equations. We now have $dw = Pd\alpha + Qd\beta$, whence it follows that w is a function of α and β.

I have shown elsewhere* that if α and β are defined by the equations

$$\begin{aligned}(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2&=c^2(t-\tau)^2,\\ l(x-\xi)+m(y-\eta)+n(z-\zeta)&=cp(t-\tau),\end{aligned}\tag{8}$$

where $\xi, \eta, \zeta, \tau, l, m, n, p$ are functions of α and β subject to the relation $l^2 + m^2 + n^2 = p^2$, then wave functions of type $\gamma f(\alpha, \beta)$ exist. It is easily seen that equations (8) are equivalent to (7) if

$$\phi = \zeta - c\tau - \theta(\xi + i\eta),\quad \psi = \zeta + c\tau + \frac{1}{\theta}(\xi - i\eta),$$

$$\theta = -\frac{l - im}{n + p} = \frac{z - \zeta - c(t-\tau)}{x - \xi + i(y-\eta)}.$$

An æthereal field may be specified with the aid of the functions α, β by defining the components of M by equations of type†

$$M_x = f(\alpha, \beta)\frac{\partial(\alpha, \beta)}{\partial(y, z)} = \frac{i}{c}f(\alpha, \beta)\frac{\partial(\alpha, \beta)}{\partial(x, t)}.\tag{9}$$

§ 5. *Model of an Element of the Æther.*

Let us now consider the case when $\tau = \alpha$ and ξ, η, ζ are functions of α only; then as Liénard has shown, the first of equations (8) will define just one real value of α if the inequalities

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* *E*, Ch. 8.

† *E*, pp. 12, 122.

$$\tau \leq t, \quad \xi'^2 + \eta'^2 + \zeta'^2 < c^2, \quad (\xi' = d\xi/d\tau),$$

are added. If, moreover, l, m, n, p are linear functions of β the second of equations (8) will define just one value of β and this will generally be a complex function of x, y, z, t. Let us choose the function $f(\alpha, \beta)$ in formulas (9) so that $f^{-1} = \beta - \beta_0$, where β_0 is an arbitrarily chosen complex quantity;* the components of M will then generally be infinite at points on the moving curve $\beta = \beta_0$.

Now when $\beta = \beta_0$ and α is kept fixed, θ has a definite complex value and so it follows that the line joining the points (x, y, z, t), (ξ, η, ζ, τ) is fixed in direction. Taking into account the first of equations (8) we see that the possible positions of (x, y, z, t) are those of a point which starts at (ξ, η, ζ, τ) and travels along a straight line with the velocity of light. By giving different real values to α we obtain a series of such points which form at any instant a singular curve of the æthereal field specified by (9). This singular curve is attached to the moving point (ξ, η, ζ, τ). If we give different complex values to β_0, a series of such curves is obtained, there being one for each value of β_0.

Let us now see if it is possible to choose the arbitrary functions at our disposal in such a way that the singular curve of an æthereal field of the type just considered is always a line of electric force in an electromagnetic field having the moving point (ξ, η, ζ, τ) as simple singularity.

Liénard has shown that an electromagnetic field with (ξ, η, ζ, τ) as a moving point charge may be derived from the potentials

$$(10) \qquad A_x = \xi'/\nu, \quad A_y = \eta'/\nu, \quad A_z = \zeta'/\nu, \quad \Phi = c/\nu,$$

where

$$\nu = \xi'(x - \xi) + \eta'(y - \eta) + \zeta'(z - \zeta) - c^2(t - \tau).$$

Writing

$$x - \xi = cl_0(t - \tau), \qquad y - \eta = cm_0(t - \tau),$$

$$x - \zeta = cn_0(t - \tau),$$

$$c - l_0\xi' - m_0\eta' - \eta_0\zeta' = \lambda, \qquad c^2 - \xi'^2 - \eta'^2 - \zeta'^2 = \mu,$$

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* It may be a function of α but not of β.

we find on calculating the components of E by means of (2) that

$$E_x = \frac{1}{cv^2}\frac{\partial\tau}{\partial t}[\mu(\xi' - cl_0) + c(t - \tau)\{\lambda\xi'' + (\xi' - cl_0)(l_0\xi'' + m_0\eta'' + n_0\zeta'')\}].$$

Now let l_0, m_0, n_0 be regarded as functions of τ and let us suppose that particles are projected from the different positions of the singularity (ξ, η, ζ, τ) in directions specified by the direction cosines (l_0, m_0, n_0), and that they travel along straight lines with the velocity of light. If (x, y, z), $(x + dx, y + dy, z + dz)$ are the coordinates at time t of two particles that were projected at times τ, $\tau + d\tau$, we find that

$$\frac{dx}{d\tau} = \xi' - cl_0 + c(t - \tau)\frac{dl_0}{d\tau}.$$

Comparing this with the expression for E_x, we see that the line joining two consecutive particles coincides in direction with a portion of a line of electric intensity at each time t, if the derivatives of l_0, m_0, n_0 are given by equations of type

$$\mu\frac{dl_0}{d\tau} = \lambda\xi'' + (\xi' - cl_0)(l_0\xi'' + m_0\eta'' + n_0\zeta'').$$

If this condition is satisfied, the series of projected particles will form at each instant t a line of electric force in the electromagnetic field with the singularity (ξ, η, ζ, τ). By taking different initial directions for (l_0, m_0, n_0) we may obtain in this way all the lines of electric force.

Since the functions l, m, n, p are arbitrary as far as their dependence on α is concerned, we may evidently choose them so that the direction given by the equation $\beta = \beta_0$ coincides at each instant τ with the direction specified by the functions (l_0, m_0, n_0). This means that we can choose a system of æthereal fields in such a way that their singular curves coincide at each instant with the lines of electric force in an electromagnetic field having (ξ, η, ζ, τ) as simple singularity.

As regards the conditions of conjugacy, we have already proved that the electromagnetic field specified by the potentials (10) is conjugate to any field of type (9) in which α, β are the special functions considered in this paragraph.* We have also proved that a field of type (9) is self-conjugate* and is such that the flow of energy (Poynting's vector) is along the radius from (ξ, η, ζ, τ) to (x, y, z, t), hence it follows that the different æthereal fields given by (9) form a mutually conjugate system.

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* E, pp. 126, 133.

There are still many questions to be answered before the present theory can be accepted. If the æther is regarded as built up of elements a law describing the mutual action of two elements is needed. The nature of the macroscopic equations satisfied by the vectors (E, H) in a region occupied by an enormous aggregate of singular curves of æthereal fields has not been ascertained. The field obtained by superposing all the æthereal fields is probably different in character from an ordinary electromagnetic field; it may possibly have some connection with the phenomena of gravitation.

JOHNS HOPKINS UNIVERSITY,
BALTIMORE,
October 27, 1914.

SHORTER NOTICES.

Handbuch des mathematischen Unterrichts. Von W. KILLING und H. HOVESTADT. Band II. Leipzig and Berlin, Teubner, 1913. x + 472 pp.

THE general character of this work was discussed by the reviewer in the BULLETIN, volume 17, No. 5. The vigor which characterized the first volume is unabated in the present one, which is devoted to trigonometry. One third of the volume is devoted to plane trigonometry, an equal amount to solid geometry or stereometry in which plane trigonometry is used freely in dealing with space problems, and the remainder to spherical trigonometry. There is sharp criticism of the ordinary text-books on trigonometry, in which the authors deplore the tendency to limit the field to goniometry and the solution of triangles with and without logarithms. Nor are they satisfied with a criticism of the content, for the methods of proof come under fire as well. As an illustration of the carefulness of their investigation, four proofs of the addition theorem for the sine as given in different texts are stated, and thoroughly discussed as to their value, not merely in proving the immediate theorem, but in suggestiveness and applicability to further problems. The author of an ordinary American text-book would not feel flattered if he measured his product by the criteria of this volume. To show the value of trigonometry as a weapon in geometry, various well-known theorems such as the Feuerbach theorem are proved and an entire chapter of 50 pages is given to the application of trigonometry to construction problems. The chapter on teaching the applications of plane trigonometry is rich in suggestions, and could be read with profit by any teacher of the subject.

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* *E*, p. 12, *Phil. Mag.*, Jan., 1914.

The section on stereometry covers a wider range than the usual elementary solid geometry. Among the applications is an interesting chapter on the geography of the sky and map making, including mercator and stereographic projection. Some chapters in fine print, such as the calculation of the surface of a curved solid by integration, and the bisection of a tetraedron by a hyperbolic paraboloid, are obviously of more interest to the teacher than to the immature student. The chapter on Euler's theorems giving the relations between the number of edges, faces, vertices, and face triangles of a convex polyedron is valuable, and shows forcibly how closely the authors have scrutinized supposed proofs. In talking about Riemann surfaces they again go beyond the depth of the ordinary student of solid geometry. The deficiency or redundancy of the numerous definitions of a regular polygon seems to be an object of keen attack. The chapter on regular spherical polygons in connection with regular polyedra is excellent and might well be incorporated into elementary texts.

The closing section on spherical trigonometry is no less critical than the earlier parts of the work, although in content it more nearly approximates a text-book on the subject. The proofs of Gauss's formulas and Napier's analogies are carefully considered, and the authors conclude their criticism of existing proofs by a new one of their own. They see no use in developing a mass of formulas which have historical interest only, so special theorems and formulas which play no rôle in the applications are reserved for the last chapter. The deduction of the theorems on the geometry of the sphere from the formulas of spherical trigonometry is a point which teachers might well note. The applications considered are the ordinary ones,—measurement of the earth, astronomical problems, determining one's position at sea, and the like. The last chapter is a discussion of the work of Möbius on spherical triangles whose sides or angles may be greater than 180°, and of the still further extensions of Study. Any reader of this volume will be impressed with the possibilities of a course in trigonometry.

D. D. LEIB.

Théorie des Nombres. Par E. CAHEN. Tome premier: *Le premier Dégré.* Paris, A. Hermann et Fils, 1914. xii + 408 pp.

THE first eight chapters contain an exposition, mainly following the ideas of Helmholtz, of the definition and properties of integers, the four fundamental operations, divisibility, greatest common divisor and least common multiple, and the theory of fractions. Chapter 9 deals with diophantine equations in one and two unknown quantities, and the euclidean algorithm. Chapters 10 to 12 contain an exhaustive treatment of diophantine equations in any number of unknown quantities, and systems of such equations, preceded by an exposition of the corresponding algebraic theories (determinants and systems of linear equations). Chapters 13 to 16 give the theory of linear substitutions and linear and bilinear forms, the number theoretic case, where the coefficients are integers, being always preceded by an exposition of the corresponding algebraic case, where the coefficients are arbitrary. Chapter 17 is concerned with linear congruences, and chapters 18 to 20 contain the fundamentals of the algebra of matrices, followed by the corresponding number theoretic propositions for matrices with integers as elements. The three last chapters give the decomposition of an integer into prime factors, the properties of Euler's function $\varphi(n)$ and some other number theoretic functions, and some remarks on linear congruences with prime modulus.

Misprints are rather numerous, though seldom misleading, and it is difficult to see why the name of Frobenius should be persistently misspelled; one may also well take exception to the narrow and unusual definition of number theory proposed in the introduction. But these are minor criticisms, and the volume should prove of great value as a text-book, since the exposition is remarkably clear and easy to follow, and illustrated by numerous and well-chosen numerical examples and exercises. Several topics, for instance the arithmetical reduction of bilinear forms, are presented here in a more lucid and accessible form than in any other work known to the reviewer.

T. H. GRONWALL.

Leçons sur la Théorie générale des Surfaces et les Applications géométriques du Calcul infinitésimal. Par GASTON DARBOUX. Première partie: *Généralités. Coordonnées curvilignes. Surfaces minima.* Deuxième édition, revue et augmentée. Paris, Gauthier-Villars, 1914. vii + 618 pp.

AFTER having been out of print for some time, the first volume of Darboux's classical treatise now appears in a second edition. Since the general features of this admirable work are undoubtedly familiar to the majority of the BULLETIN'S readers, it will be sufficient to mention, in this review, the new matter added in the second edition and taken mostly from Darboux's own papers in the *Bulletin des Sciences mathématiques* and the *Comptes rendus*.

In Book I, Chapters 5 and 6 deal with the kinematics of motions dependent on any number of parameters, and the integration of the corresponding total differential equations for the direction cosines. Chapter 7, dealing with the special case of two parameters, brings some new developments regarding the Plücker conoid. Chapter 10 gives a new solution, which is both elementary and elegant, of a problem first proposed and solved by Sophus Lie: to determine all surfaces which can be generated in more than one way by the translation of a rigid curve. It seems, however, to have escaped the notice of the distinguished author that a similar solution has been previously given by Scheffers ("Das Abelsche Theorem und das Liesche Theorem über Translationsflächen," *Acta Mathematica*, volume 28 (1904), pages 65–91).

In Book II, there has been added to Chapter 1 a study of a special conjugate system formed by plane curves, which leads to a class of surfaces applicable on quadric surfaces and discovered by Peterson. Chapter 4 contains an exposition of Gauss's method for the conformal representation of the terrestrial ellipsoid on a sphere, as well as a solution of a problem proposed by Lagrange: to determine all geographical maps such that the meridians and parallels are represented by circles.

In Book III, Chapter 3 gives the determination of all minimal surfaces capable of being generated by translation in more than one way, and in Chapter 8 a special minimal surface is considered which was discovered by Riemann and is generated by the motion of a circle of variable radius. Finally, Chapter 9 gives an investigation of Ribaucour's isotropic congruences, with applications to Bertrand's curves.

T. H. GRONWALL.

NOTES.

THE opening (January) number of volume sixteen of the *Transactions of the American Mathematical Society* contains the following papers: "On the theory of curved surfaces, and canonical systems in projective differential geometry," by G. M. GREEN; "The multitude of triad systems on 31 letters," by H. S. WHITE; "The φ-subgroup of a group," by G. A. MILLER; "On a set of postulates which suffice to define a number-plane," by R. L. MOORE; "The equivalence of complex points, planes, lines with respect to real motions and certain other groups of real transformations," by W. C. GRAUSTEIN; "Invariants of the rational plane quintic curve and of any rational curve of odd order," by J. E. ROWE; "A set of postulates for general projective geometry," by M. G. GABA; "Certain quartic surfaces belonging to infinite discontinuous cremonian groups," by V. SNYDER and F. R. SHARPE; "The functions of a complex variable defined by an ordinary differential equation of the first order and the first degree," by J. SLEPIAN; "On the differential geometry of ruled surfaces in 4-space and cyclic surfaces in 3-space," by A. RANUM.

THE opening (January) number of volume 37 of the *American Journal of Mathematics* contains the following papers: "Generalizations of geodesic curvature and a theorem of Gauss concerning geodesic triangles," by G. A. BLISS; "On the medians of a closed convex polygon," by A. EMCH; "The rational sextic curve, and the Cayley symmetroid," by J. R. CONNER; "Limited and illimited linear difference equations of the second order with periodic coefficients," by T. FORT; "Finite groups of plane birational transformations with eight fundamental points," by F. R. SHARPE; "Conics through inflections of self-projective quartics," by F. R. SHARPE; "Concerning an analogy between formal modular invariants and the class of algebraical invariants called Booleans," by O. E. GLENN; "Periodic orbits on a smooth surface," by D. BUCHANAN; "On Foucault's pendulum," by W. D. MACMILLAN.

The frontispiece of the volume is a portrait of JOHN NAPIER.

THE first number (January 15, 1915) of the *Proceedings of the National Academy of Sciences* contains the following mathematical papers: "Recent progress in the theories of modular and formal invariants and in modular geometry," by L. E. DICKSON; "The synthesis of triad systems Δ_t in t elements, in particular for $t = 31$," by H. S. WHITE; "The φ-subgroup of a group of finite order," by G. A. MILLER.

The second number (February 15) contains: "Transformations of surfaces Ω," by L. P. EISENHART; "Conjugate systems of space curves with equal Laplace-Darboux invariants," by E. J. WILCZYNSKI.

THE following papers on mathematics or mathematical physics have recently appeared in the *Proceedings of the American Academy of Arts and Sciences:* "The generalized Riemann problem for linear differential equations, and the allied problems for linear difference, and q-difference equations," by G. D. BIRKHOFF, volume 49, pages 519–568; "On the theory of the rectilinear oscillator," by E. B. WILSON, volume 50, pages 105–128; "Planck's radiation formula and the classical electrodynamics," by D. L. WEBSTER, volume 50, pages 128–145.

AT the meeting of the London mathematical society held December 10 the following papers were read: "Simultaneous equations, linear or functional," by E. H. NEVILLE; "Cyclotomic quinquisection," by W. BURNSIDE; "Oscillations near the isosceles triangle solution of the three body problem," by D. BUCHANAN; "Lamé's differential equation and ellipsoidal harmonics," by E. T. WHITTAKER.

THE Mathematical association held its annual meeting January 9 at the London day college. The following papers were read: "Mathematics in artillery science," by G. GREENHILL; "Teaching of modern analysis," by W. P. MILNE; "The circle of curvature," by A. LODGE; "Practical work in connection with mathematics," by R. C. FAWDRY.

A NOTICE has recently been sent out by Professor Fehr, secretary of the International commission on the teaching of mathematics, announcing that, on account of the war, it has been decided not only to abandon the idea of a meeting in August 1915, but also to postpone the preparation of such projected reports as relate to the European countries.

THE University of Chicago Press announces the following volumes in preparation for early publication in its Science Series: Finite collineation groups, by H. F. BLICHFELDT; Linear integral equations in general analysis, by E. H. MOORE.

AT Harvard University two Benjamin Peirce instructorships in mathematics have been established, the holders of which will devote a part of their time to teaching, including some advanced instruction, but will be given facilities and encouragement to prosecute scholarly research. Appointments are made, under open competition, for a term of one year, and may be renewed for not more than three years. Further information may be obtained from the chairman of the division of mathematics, PROFESSOR BÔCHER.

AT Cornell University provision has been made for giving two cycles of advanced courses in analysis and in geometry in the summer session, each cycle to run through four years. For 1915 these courses will begin with Advanced calculus and infinite series, by Professor W. A. HURWITZ, five hours, and Algebraic plane curves, by Professor VIRGIL SNYDER, five hours.

PROFESSOR DE LA VALLÉE POUSSIN, of the University of Louvain, will give a course of lectures in French at Harvard University on Lebesgue integrals, beginning February 16 and extending through the remainder of the academic year.

DR. M. BOTTASSO has been appointed docent in analytic geometry at the University of Pavia.

DR. T. H. HAVELOCK has been appointed professor of applied mathematics and mathematical physics at Armstrong College, Newcastle-on-Tyne.

PROFESSOR A. W. PHILLIPS, formerly dean of the Yale University graduate school, died January 20, at the age of 70 years. He became a tutor at Yale in 1877 and retired from his professorship in 1911. He was a member of the American Mathematical Society since 1896.

PROFESSOR A. VENTURI, of the University of Palermo, died December 29 at the age of 62 years.

Book catalogues: Deighton, Bell and Company, 13 Trinity Street, Cambridge, England, catalogue 27, some 300 mathematical pamphlets, theses, etc.—Gustav Fock, Schlossgasse 7, Leipzig, Handapparate of Amthor, Lüroth, Schubert, and others.—Julius Springer, Linkstrasse 23, Berlin, technical works, applied mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BAYLISS (R. W.). A first school calculus. New York, Longmans, 1914. Cr. 8vo. 12+288 pp. $1.20

BLUEL (E.). Leçons de mathématiques spéciales. Volume 1: Algèbre, ligne droite et plan, trigonométrie, analyse, applications géométriques. Paris, 1914. 8vo. Fr. 15.00

——. Leçons de mathématiques spéciales. Volume 2: Géométrie analytique, courbes et surfaces. Paris, 1914. 8vo. Fr. 15.00

BÔCHER (M.). Charles Sturm et les mathématiques modernes. Paris, Alcan, 1914. 8vo. 19 pp.

—— Mathématiques et mathématiciens français. Paris, Champion, 1914. 8vo. 16 pp.

BOREL (E.). Leçons sur la théorie des fonctions. Eléments et principes de la théorie des ensembles; applications à la théorie des fonctions. 2e édition. Paris, Gauthier-Villars, 1914. 8vo. 12+260 pp. Fr. 7.50

CREMONA (L.). Elements of projective geometry. Translated by Charles Leudesdorf. 3d edition. Oxford, Clarendon Press, 1914. 8vo. 322 pp. 12s. 6d.

CZUBER (E.). Wahrscheinlichkeitsrechnung und ihre Anwendung auf Fehlerausgleichung, Statistik und Lebensversicherung. 3te erweiterte Auflage. (2 Bände.) Band I: Wahrscheinlichkeitstheorie, Fehlerausgleichung, Kollektivmasslehre. Leipzig, Teubner, 1914. Gr. 8vo. 12+462 pp. Cloth. M. 14.00

HJELMSLEV (J.). Grundlag for flaudernes geometrie. Kjöbenhavn, Vid. Selsk. Skrift., 1914. 8vo. 52 pp.

HORT (W.). Die Differentialgleichungen des Ingenieurs. Berlin, Springer, 1914. 8vo. 9+540 pp. Cloth. M. 14.00

KOMMERELL (V.). Raumgeometrie (Stereometrie und darstellende Geometrie). 2te Auflage. Tubingen, H. Lauppsche Buchhandlung, 1914. 213 pp. M. 2.80

LEUDESDORF (C.). See CREMONA (L.).

LOEWY (A.). Lehrbuch der Algebra 2 Theile. Theil I: Grundlagen der Arithmetik. Leipzig, 1914. Gr. 8vo. M. 12.00

LOUP (W. M.). Bijdrage tot de theorie der cyclische oppervlakken en congruenties. Utrecht, 1914. 8vo. 88 pp.

MATHEMATISCHE ABHANDLUNGEN Hermann Amandus Schwarz zu seinem funfzigjahrigen Doktorjubiläum am 6. August 1914 gewidmet von Freunden und Schülern. Berlin, Springer, 1914. 8vo. 8+451 pp. 53 figures. M. 24.00

MATHEWS (G. B.). Projective geometry. (Longmans Modern Mathematical Series). London, Longmans, 1914. 8vo. 14+349 pp. 5s.

PALM (R.). Zur Invariantentheorie eines Pfaffschen Ausdrucks. (Diss., Greifswald.) Leipzig, Teubner, 1914.

ROBB (A. A.). A theory of time and space. Cambridge, University Press, 1914. Royal 8vo. 6+373 pp. Cloth. 10s. 6d.

SCHATUNOVSKY (J.). Der grösste gemeinschaftliche Teiler von algebraischen Zahlen zweiter Ordnung mit negativer Diskriminante und die Zerlegung dieser Zahlen in Primfaktoren. Strassburg, 1912. 8vo. 67 pp.

SCHWARZ (H. A.). See MATHEMATISCHE ABHANDLUNGEN.

II. ELEMENTARY MATHEMATICS.

BARNARD (S.) and CHILD (J. M.). Elements of geometry. Parts 1–6. London, Macmillan, 1914. 8vo. 9+465+38 pp.+answers. 4s. 6d.

ČERVENKA (L.). See VOROVKA (K.).

CHILD (J. M.). See BARNARD (S.).

COOK (F. W.). See WILKINSON (P.).

CRAWLEY (E. S.). One thousand exercises in plane and spherical trigonometry. Philadelphia, E. S. Crawley, 1914. 8vo. 70 pp.

DANZER (O.). Die darstellende Geometrie am Realgymnasium. Wien, 1912. 8vo. 63 pp.

DOLINSKI (M.). Politische Arithmetik. Zinseszinsenrechnung und Versicherungsmathematik nebst zwei Logarithmentafeln. Wien, 1914. Gr. 8vo. 292+57 pp. M. 5.00

DUPUIS (J.). Tables de logarithmes à 7 décimales, contenant les logarithmes des nombres de 1 à 100,000 et des sinus et tangentes des angles calculés de seconde à seconde pour les 5 premiers degrès et de 10 secondes à 10 secondes pour tous les degrès du quart cercle. 46e tirage. Paris, 1914. 8vo. 11+581 pp. Fr. 9.00

ERIKSSON (E. V.). Om sectio anguli och om de af detta problem betingade algebraiska ekvationera. Göteborg, 1914. 8vo. 41 pp.+3 tables. M. 1.50

GILLESPIE (F. C.). Worked exercises in elementary geometry. Oxford, Blackwell, 1914. 8vo. 4+136 pp. 3s. 6d.

GUEZOUNI (C.). The trisection of lines and angles. London, International Scientific Association, 1914. 1s.

HOWELL (H. B.). A foundation study in the pedagogy of arithmetic. London, Macmillan, 1914. 5s. 6d.

JÄDERIN (E.). Tables abrégées pour le calcul des logarithmes à 12 décimales. Stockholm, 1914. 4to. 21 pp.

LONGMANS explicit arithmetics. London, Longmans, 1914. 8vo. Pupil's Book IV, 3d. Teacher's Book IV, 1s.

POSEJPOL (V.). See VOROVKA (K.).

RICHARDSON (S. F.). Solid geometry. Boston, Ginn,. 1914. 8vo. 5+209 pp. $0.90

SAMSONOFF (J.). The trisection of an angle. New York, Publishers Printing Co., 1914. 8vo. 12 pp. $0.25

SANJANA (K. J.). First year course in mathematics. Geometry and trigonometry. Bombay, K. & J. Cooper, 1914. 240 pp.+15 tables. Ru. 1.12

SANKEY (E.). Examples in easy practical drawing for the second year preliminary technical course. Books 1–2 with answers. 66+66 pp. 8d.+8d.

SMITH (D. E.). See WENTWORTH (G.).

SUZZALO (H.). See WALSH (J. H.).

VOROVKA (K.), ČERVENKA (L.) und POSEJPOL (V.). Die Lehrbücher für Mathematik, darstellende Geometrie und Physik an den Mittelschulen mit bohmischer Unterrichtssprache. (Bericht uber den math. Unterricht in Oesterreich.) Wien, Holder, 1914. 89 pp.

WALKER (W. J.). Examples and test papers in algebra. Parts 2–3. London, Mills and Boon, 1914. 8vo. 8+163–338 pp. 1s. 6d.

WALSH (J. H.) and SUZZALO (H.). Three-book series of arithmetics. Boston, Heath, 1914. 12mo. Cloth. Fundamental processes. 6+246 pp. $0.35. Essentials. 6+290 pp. $0.40. Business and industrial practice. 9+366 pp. $0.45.

——. Two-book series of arithmetics. Boston, Heath, 1914. 12mo. Cloth. Fundamental processes. 6+246 pp. $0.35. Practical applications. 6+502 pp. $0.65.

WENTWORTH (G.) and SMITH (D. E.). Plane trigonometry and tables. Boston, Ginn, 1914. Royal 8vo. 6+188+6+104 pp. $1.10

WILKINSON (P.) and COOK (F. W.). Macmillan's reform arithmetic for rural schools. (Standards V to VII). London, Macmillan, 1914. 5d.

YATE-LEE (F. S.). An introductory geometry. Part 1 (for teachers only). 19 pp. Part 2. 32 pp. Ripon, Thirlway, 1914.

III. APPLIED MATHEMATICS.

BAUERMANN (E.). Untersuchungen über die Ausführbarkeit einer Methode, Kontaktpotentialdifferenzen durch ein trennendes Dielektrikum hindurch zu messen. Rostock, 1912. 8vo. 37 pp. M. 1.50

Bouasse (H.). Cours de physique. 2e édition, complètement transformée et augmentée. Tome III: Magnétisme et électricité (3 parties). Partie 2: Applications. Paris, Delagrave, 1914. 8vo. 560 pp. Fr. 22.00

Bonin (H.). Experimentelle Bestimmung des Ungleichsförmigkeitsgrades. (Forschungsarbeiten auf dem Gebiete des Ingenieurwesens, herausgegeben vom Verein Deutscher Ingenieure, Heft 165). Berlin, 1914. Lex. 8vo. 35 pp. M. 1.00.

Brunner (J. B.). Lehrbuch der Physik. Zürich, 1914. 8vo. 420 pp. +4 tables. Geb. M. 7.30

Bürgin (J.). Genauigkeitsuntersuchungen über die Bestimmung der Intensität der Schwerkraft durch relative Pendelmessungen auf 9 Stationen des Badischen Oberlandes und auf der Schweizerischen Referenzstation zu Basel. Heidelberg, 1912. 4to. 112 pp.+6 tables.

Carte photographique du ciel (Observatoire de Toulouse). Coordonnées rectilignes. Tome IV: Zone +6° à +8°. Fascicule 2. Paris, 1913. 4to. Pp. 53–220. Fr. 23.00

Crabtree (H.). An elementary treatment of the theory of spinning tops. London, Longmans, 1914. 8vo. 15+193 pp. 7s. 6d.

Flamm (L.). Die neuen Anschauungen über Raum und Zeit. Das Relativitatsprinzip. Wien, Schrift. Ver. Verbr. nat. Kenntn., 1914. 8vo. 46 pp.

Flammarion (C.). Le système du monde et Copernic. Nouvelle édition. Paris, 1914. 8vo. Fr. 0.60

Fowle (F. E.). Smithsonian physical tables. 6th revised edition. (*Smithsonian Miscellaneous Collections*, Vol. 63, No. 6). Washington, Smithsonian, 1914. 36+355 pp.

Gibson (A. H.) and Ritchie (E. G.). A study of the circular-arc bow-girder. London, Constable, 1914. 4to. 10s. 6d.

Graetz (L.). Die Elektrizität und ihre Anwendungen. 17te umgearbeitete und vermehrte Auflage. Stuttgart, 1914. Gr. 8vo. 16+748 pp. Cloth. M. 9.00

Graff (K.). Grundriss der geographischen Ortsbestimmung aus astronomischen Beobachtungen. Berlin, 1914. Gr. 8vo. 8+210 pp. M. 8.00

Greenwich. Astronomical, magnetical and meteorological observations made at the Royal astronomical observatory, Greenwich, in the year 1912. London, 1914. Royal 4to. 21s.

Haas (A. E.). Die Grundgleichungen der Mechanik, dargestellt auf Grund der geschichtlichen Entwicklung. Vorlesungen zur Einfuhrung in die theoretische Physik. Leipzig, 1914. Gr. 8vo. M. 7.50

Haler (P. J.) and Stuart (A. H.). First course in mathematics for technical students. London, Tutorial Press, 1914. 1s. 6d.

Haussner (R.). Darstellende Geometrie. 1ter Teil, 3te Auflage. 207 pp. 2ter Teil, 2te Auflage. 168 pp. (Sammlung Göschen Nr. 142 u. 143.) Leipzig, Goschen, 1914. M. 0.90+0.90

Havelock (T. H.). The propagation of disturbances in dispersive media. (Cambridge Tracts in Mathematics, No. 17.) Cambridge, University Press, 1914. 8vo. 3s. 6d.

Hobb (—.). Hobb's arithmetic of electrical measurements. Revised by A. Risdon Palmer. London, Murby, 1914. 1s.

HOUSDEN (C. E.). Rapid earthwork calculation. London, Longmans, 1914. Cr. 8vo. 31 pp. 2s.

IKLÉ (M.). See MCCREADY (K.).

INGHAM (A. E.). Machine construction and drawing. Book 2. London, Routledge, 1914. 12+180 pp. 3s.

KOLLERT (J.). Die Wechselstrommaschinen, Transformatoren und Motoren. Kurze Darstellung ihres Baues und ihrer Wirkungsweise. 2te erweiterte Auflage von "Riedel, Wechselstrommaschinen." Leipzig, 1914. Gr. 8vo. 7+159 pp. M. 5.00

LOW (D. A.). An introduction to machine drawing and design. New edition, rewritten, reillustrated and enlarged. London, Longmans, 1914. 8vo. 8+248 pp. 3s.

MCCREADY (K.). Sternbuch für Anfänger. Uebersetzt von Max Iklé. 8vo. Leipzig, Barth, 1913. 9+150 pp.+2 tables. Geb. M. 12.00

MARCHANT (T. P.). See MORGAN (J.).

MORGAN (J.), MARCHANT (T. P.) and WOOD (A. L.). The "Conway" manual, being a complete summary of all problems of navigation and nautical astronomy. London, Potter, 1914. 79 pp. 5s.

MÜLLER (O.). Graphisches Rechnen und die graphische Darstellung. Ausgabe B. Glauchau i. S., Streit, 1913. 64 pp. Geb. M. 2.00

PALMER (A R.). See HOBB.

PLANCK (M.). Dynamische und statische Gesetzmässigkeit. Leipzig, 1914. 8vo. 32 pp. M. 1.00

RIEDEL (—). See KOLLERT (J.).

RIEGLER (G.). Sonnen- und Mondfinsternisse und ihre Bedeutung für die Himmelsforschung, leichtfasslich dargestellt. Wien, 1914. 8vo. M. 2.00

RITCHIE (E. G.). See GIBSON (A. H.).

SPOONER (H. J.). Engineering workshop drawing. First steps in engineering drawing. London, Longmans, 1914. Oblong 4to. 8+120 pp. 2s.

STUART (A. H.). See HALER (A. J.).

TURNER (H. H.). Tables for facilitating the use of harmonic analysis. Oxford, Clarendon Press, 1913. 1s.

VILLAMIL (R. DE). Motion of liquids. London, Spon, 1914. 8vo. 14+210 pp. Cloth. 7s. 6d.

WALKER (T. L.). Crystallography: an outline of the geometrical properties of crystals. New York, McGraw-Hill, 1914. (London, Hill Publ. Co.) 8s. 4d.

WOOD (A. L.). See MORGAN (J.)

L R RY

THE RÔLE OF THE POINT-SET THEORY IN GEOMETRY AND DYNAMICS.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY, JANUARY 1, 1915.

BY PRESIDENT E. B. VAN VLECK.

IN his "Child's Garden of Verses" Robert Louis Stevenson says simply but poetically:

> "The world is so full of a number of things,
> I'm sure we should all be as happy as kings."

This catches the spirit of the Mengenlehre, and may well be taken as its motto. In more homely phrase the mathematician and physicist declare that the world is made up of an *infinite* number of things. Hence in thought and nature we have to consider numbers which are infinite. Even though they be not actually infinite, they are so enormously, unthinkably large that they can be most advantageously handled by regarding them as infinite. Such a mode of treatment was, of course, foreign to Greek thought. But, as you all know, the classic Greek mathematics was straitened by its failure to admit the infinitesimally small and the infinitely great magnitudes of the calculus. In somewhat similar fashion modern mathematics was long hampered by the lack of a mathematical theory, not of the potentially infinite quantity of the calculus, but of actually infinite aggregates. It was the incalculable and distinctive service of Georg Cantor to have perceived the urgent need of such a theory. Minerva-like and full panoplied it sprang from his teeming brain. Already its achievements have been very great, but it is far from maturity and its full powers are still to be revealed.

The influence of this new theory of infinite aggregates was first decisively felt in the theory of functions of the real and complex variables, where infinite sets of points, often irregular in character, frequently present themselves; for example, the discontinuities of an integrable function or the singularities of an analytic function, either of which may occur in infinite number. Here also, in the function-theory, its richest fruition has been attained. Three stages in the history of the Mengenlehre have been distinguished by Borel.* At first an offshoot or branch of the function theory, it soon reached a stage of independent existence where it was studied by and for itself. But its methods and ideas so rapidly permeated the function-theory that a third period is distinguished by Borel, in which its essentials have been re-absorbed into the function-theory. The time would appear to be opportune for a general view of their mutual interaction, but inasmuch as this is so well-known in many of its aspects,† I have preferred to consider other bearings of Cantor's theory.

Quite in contrast with the definitive effects in the function-theories is the plastic and vitalizing action of the Mengenlehre upon logic and philosophy. This may well be considered as beginning a fourth period, formally inaugurated at Paris last April in the first international congress for mathematical philosophy. Considered abstractly, Cantor's theory led inevitably to questions regarding the modes and soundness of reasoning upon infinite aggregates. Certain well-known antinomies resulted, demanding imperatively a solution. This has been generally sought by abandoning Cantor's definition of a set‡ as any collection of definite and well-separated objects of perception or thought. For how otherwise can we make sure, as we certainly must, that there is no contradiction lurking in the set of objects brought together? By what, then, shall Cantor's definition be replaced? Must we confine ourselves to sets of objects each of which has been clearly defined in a finite number of words? The child begins to think on the world around him, undeterred by the consideration that the number of objects is limitless while he can reason in only a finite number of words. Numberless errors are made by him, the discovery and correction of which enhance his knowledge and power. Likewise in the early renaissance Cavalieri, Kepler, and others argued most imperfectly on indivisibles and infinites. But out of their thinking sprang ultimately the most powerful branch of mathematics. Undoubtedly great dangers lie hidden in reasoning on infinite classes of distinct objects whose elements have not all been defined and separated from one another by a finite number of words—for example, the points of a continuum. Principles may be used which subsequently will be discovered to be false, or at least more restrictedly true than had been imagined. But is this a sufficient cause for abandoning the so-called idealistic reasoning? Is it not rather a true mode of discovery? By boldly plunging forward in idealistic reasoning shall we not avoid the danger of falling into sterile rules of thought? Or, to the contrary, shall we thereby merely stumble into the pit we are trying to avoid? Is the Princip der Auswahl sound, or mere sound? Does mathematical reasoning involve principles which cannot be embraçed in the syllogism, as claimed by Poincaré and denied by Russell; for instance, the principle of mathematical induction? Such questions as these, and many others, are merely an indication of the fermentation started by the Mengenlehre which has resulted in the re-creation of a department of mathematical logic and philosophy.

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* Cf. *Revue générale des Sciences*, vol. 20 (1909), p. 315.

† Various aspects are well presented by Borel; loc. cit.

‡ "Unter einer Menge verstehen wir jede Zusammenfassung von bestimmten wohlunterschiedenen Objekten unserer Anschauung oder unseres Denkens zu einem Ganzen." *Math. Ann.*, vol. 46 (1895), p. 481.

Into the consideration of this new and potent, but most abstract influence of the theory of aggregates I also do not propose to enter. My interest today lies neither in its best nor least formulated influence. It was the expectation of both Cantor and du Bois-Reymond that the theory of point-sets would become very useful in the field of mathematical physics. This expectation has been so little realized that even to the present time it is commonly said that no applications are there to be found. Though, indeed, as yet the applications to astronomy and physics are but few, they seem to me of great significance and promise. In fact, dynamics must always be deeply dependent for its development upon geometry, since dynamical trajectories are special cases of geometric curves. It seems to me, therefore, fitting to consider simultaneously: The service of the point-set theory to geometry and dynamics.

The rôle of any great branch of mathematics may be due to one or more of three causes,—its fundamental concepts, its distinctive methods, or its principal results. In the case of the point-set theory the rôle has been determined chiefly by its concepts and methods, which are intimately related. The greatness of a mathematical method or concept—for example, the notion of a group or of a differential coefficient—may be tested by its simplicity and universality. As has been so often said, the human intellect demands economy and convenience of thought. The scientist selects such principles as fulfil this demand and calls them true and fundamental. Not less imperative is it in mathematics to search for those concepts, methods, and theorems which are simple and widely applicable. This, let me urge parenthetically, is a lesson we may well ponder if our ambition for the greatness of American mathematics is to be attained. Judged by the same standard, the point-set theory is strong in both its methods and concepts, but preeminently in the latter.

To Georg Cantor was granted the rare fortune and merit to have uncovered more than one great concept. Some of his notions, as for instance the "power" of a set and the transfinite ordinal number of the first class, have as yet found their application almost wholly within the function-theory, and these I shall have no occasion to introduce. The concepts which I shall need are only five in number. Two of these are the familiar notions "limit point" and "everywhere dense," which antedate the work of Cantor. It was Cantor, however, who focused the search light upon the totality of the limit points of a given set, and called this totality the "derivative" or "first derived set" of the given set. The fourth concept is that of a "perfect set," which results when a set coincides with its derivative, so that it contains all its limit points and nothing but these. The fifth and last notion is perhaps Cantor's greatest concept, that of a "countable" or "enumerable" set, with which every one is familiar.* In particular, he showed the set of algebraic numbers to be countable.

On this concept of countability is based the modern analysis of the continuous interval, which lies at the bottom of ordinary geometry, be it euclidean or non-euclidean, two-dimensional or n-dimensional. Every well-informed mathematician knows how at last a satisfactory foundation for a continuous number system was discovered by Dedekind. His memoir of 1872 in which this was published is contemporary with Cantor's earliest publications relating to the Mengenlehre and may worthily be regarded with Cantor's work as the chief spring of our point-set theory. Starting with the system of rational numbers arranged in natural order of magnitude, Dedekind considered the cuts through the system which divide it into two parts, lying to the right and left of the cut respectively. Wherever there was a "gap" caused by the cut—i. e., neither a last element before the cut nor a first element after the cut—he filled in the gap with an irrational number. A definite type of order (the "order-type" θ) is thus obtained for the continuous number system created. In accordance with this analysis a continuous interval becomes merely an ordered aggregate of separate points, non-countable in number because the gaps in the system of rational points in their natural order were shown by Cantor to be non-countable. Hence the mathematical continuum or matrix—and, if you so will, the physical ether to correspond—is resolved into separate but ordered constituents with a reality and discreteness comparable to that of a row of marbles extending infinitely far to right and left.

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* A set is said to be countable when by proper selection we can count its elements, "one, two, three," etc., ad infinitum, omitting no element and reaching each element at some time in our count.

It is worthy of remark that this "arithmetization" of the continuous interval is in everyway parallel to the recent progress of physics, which tends more and more to become a theory of discrete molecules, atoms, and electrons. Even the existence of the matrix or ether in which these are imbedded has become problematical and open to suspicion. To the atomic structure of a continuous region, if I may call it that, the mathematician also has been driven by the necessity for clear thinking. The question whether anything can be gained for physical theory by supposing our continuous space or ether to be characterized by a higher and more complex order-type has, so far as I know, not been even broached. Within our present continuous space the point-set theory enables us increasingly to consider different groupings of points. In his Houston Lectures Borel has remarked how different physical theories require and engender different mathematical theories. Thus the differential calculus tallies with a physical theory of continuous matter. It seems by no means a wild fancy that because of its essentially atomic structure the point-set theory may develop into an indispensable tool of an atomic physics.

The analysis of the continuum is closely connected with one of the characteristics of modern geometrical development, the search for an axiomatic basis. While too much should certainly not be claimed for the point-set theory in this regard, it is probably not too much to say that it has given, in a general rather than a specific way, a strong impulse to this search, though, of course, one entirely secondary to the impulse imparted by the non-euclidean geometry. Yet even more than the latter, it involves a most careful critique of geometrical relations. Historically considered, it is difficult to believe that Hilbert's Foundations of Geometry could have been written before a considerable development of point-set theory. Be this as it may, the rôle of the theory becomes clear and explicit in one of Hilbert's later communications* on the Foundations of Geometry which lays the foundation for geometrical motion.

In this memoir Hilbert achieved the solution of a problem of non-euclidean geometry proposed long before by Riemann, who (with Helmholtz) was the first to attack this subject upon the analytical side. Taking space as the "Zahlenmannigfaltigkeit" of analytical geometry, Riemann asked what properties must be imposed upon this number-spread to give it the character of an ordinary geometry, though not necessarily a euclidean one. His fundamental concept and tool was the element of arc, to which his axioms relate. But to build a finite geometry upon the element of arc, integration is necessary.† Now neither the notion of integration nor of an element of arc is fundamentally simple. Hence his solution, as also for other reasons, cannot be regarded as satisfactory. Later the problem was attacked by Lie, whose attention had been called by Klein to the fact that the notion of a continuous group was involved in both Riemann's and Helmholtz's work on non-euclidean geometry. Lie's axiomatic handling of the problem by the use of the continuous group is also very far from elementary, since in defining the group he employs only transformations which are represented by differentiable functions. Also for the same reason his results are restricted in generality.

Such was the condition of the problem when it was attacked in 1902 by Hilbert. His communication is in his best style and has the elegance always characteristic of his geometrical work. He lays down just three axioms of motion, which I shall not stop to explain but which may be expressed tersely but vaguely as follows:

1. The movements of the plane into itself form a group.

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* *Math. Ann.*, vol. 56 (1902), p. 381.

† Cf. Lie-Engel, Theorie der Transformationsgruppen, vol. 3, p. 395.

2. Every ("wahre") circle contains infinitely many points.
3. The movements form a closed system.

A geometry based on these axioms is then proved to possess the character of either the euclidean geometry or the hyperbolic geometry of Bolyai and Lobachevsky. In the proof Hilbert sloughs off entirely such unnecessary elements as differentiability and even the employment of functions.

The memoir is a splendid interweaving of the basal concepts of two great subjects, the group and point-set theories. From almost the first line to the last it is saturated with the notions and methods of the point-set theory. I know of no more beautiful application of the theory to be found anywhere. I have chosen it therefore as a peg on which to hang some of my remarks.

First of all, I wish you to notice that the point-set theory must enter indispensably into problems like this, which deal with the foundations of geometry and seek the most general conditions under which our space theorems and assumptions are to hold. For when the axiomatization of geometry passes beyond its simpler stages, we must begin to consider infinite aggregates of points, duly grouped and arranged. Hence point-set theory implicitly or explicitly begins to enter. We have not merely to decompose a theorem or assumption into simpler fundamental elements and relationships, but in the synthesis which follows we have also to combine into sets all points or curves which fulfil certain conditions. Thus when Hilbert considers all possible motions of a plane into itself which leave a particular point unaltered, he must show that the different positions taken by any other point form a set of points infinitely dense in themselves and closed, in other words, a perfect set; that this perfect set has the order type of the linear continuum, consisting of an ordered countable set of points dense in itself and their limit points; that this linearly ordered set is a closed Jordan curve containing in its interior the fixed point of the motion; that these Jordan curves of motion enclose one another like the set of concentric circles; and so on. In this manner he demonstrates that in any motion of the plane which leaves a fixed point invariant, the path described by any point possesses all the essential characteristics of a circle viewed in the light of analysis situs.

Furthermore, Hilbert's problem is typical of an ever increasing number of problems in which we have to settle the exact geometrical character of a set of points, the locus being often constructed point by point through some analytic process with a final combination of them into a whole. Frequently the condensation method is applied, of which I shall speak later. The problems may differ widely in character, ranging from the consideration of such abstract loci as Hilbert here obtains to the determination of the nature of a dynamical trajectory or of the set of singularities of an analytic function.

A third aspect of Hilbert's work which I wish to emphasize is that it belongs to the province of analysis situs. The plane with which he is dealing is defined as a system of points or "Dingen" which can be referred by a one-one correspondence to the finite number plane or a portion (Teilsystem) of the same. Some of you have doubtless been already asking what is to be understood by the term "movement of a plane" into itself which I have been using. By Hilbert's definition this is any continuous 1–1 transformation of the number image of the plane into itself which leaves unaltered the sense of a circuit around any Jordan curve of this image-plane. Hence by very definition the problem is one of analysis situs. The increasing importance of this branch of mathematics need scarcely be argued here. Ever since Riemann revealed the importance of the "genus" or "class" for a Riemann surface and for the corresponding functions, the qualitative distinctions of analysis situs have proved to be fundamental for the detection of differences in analytic or geometric theories and for their differentiation one from another. Now it is this province of mathematics which is today being invaded by the two allies, the function-theory and the theory of point-sets, and so successfully has it been invaded that its independence has been almost lost.* It remains only to determine which of the two allies will annex it or how they will divide the spoils. In his report at the Zurich International Congress (1897) on the "Development of the general theory of analytic functions in recent time" Hurwitz stated "die Aufgabe der Analysis Situs" to be this, "die Invarianten der einzelnen Klassen von Punktmengen auszusuchen." I wish to preserve my neutrality, but I venture nevertheless to interpret this as an early prediction of the annexation of analysis situs by the theory of point-sets. As examples of invariants of analysis situs which are unaltered by any continuous 1–1 point transformation, I may cite such concepts as limit-point, a perfect point-set, the power of a point-set, and the accessibility or inaccessibility of a specified point of the boundary of a region by means of a path from any interior point which, save for its terminal boundary point, lies wholly in the interior of the region. Examples of invariantive properties are expressed in the familiar theorem that a Jordan curve divides the plane into two regions, also the theorem of Brouwer that the number of dimensions of a space is unaltered by a continuous 1–1 transformation; and so on.

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* However, one large division of the subject has been scarcely affected by the methods or ideas of point-set theory.

The enrichment of geometry by the theory of point-sets is by no means confined to the department of analysis situs. The new theory has shown itself strong—I might almost say unexpectedly strong—on the quantitative as well as on the qualitative side. As early as 1882 an effort was made by Hankel to attribute content to a discrete set of points. This concept contributed a new element to the growth of the theory. In the more comprehensive form of "measure" devised by Borel and Lebesgue it becomes a powerful agent for quantitative discrimination between sets of points. Thus the rational points in an interval are given a measure 0 and in consequence become negligible in comparison with the irrational points, whose total measure is equal to the length of the interval. With this keen-cutting instrument, measure, Lebesgue has sharpened Riemann's definition of an integral or, more exactly stated, has fashioned a new integral to take its place, far more comprehensive and for theoretical purposes much more powerful. It is true that the new integral has been little applied as yet in geometry, but its usefulness in analysis leaves little doubt that it must eventually become important in geometry also. Such intuitive ideas as the length of a curve or the area of a surface involve much difficulty as soon as we leave the simpler cases, and require the use of measure or other point-set analysis for their development. By point-set means various species of curves have been distinguished, such as Jordan curves, rectifiable and non-rectifiable curves, space-filling curves, the Lebesgue-Osgood curves with areal content, and so on. I may also mention Lebesgue's developable surfaces which are not applicable to a plane, and applicable surfaces which are non-developable. These specifications will suffice to indicate the enrichment of geometry through new ideas. A large number of the familiar geometrical concepts, such as the instantaneous curvature of a curve or a surface, were brought in through the differential calculus and begot a differential geometry. Is it too much to expect that new ideas of a totally different import will cause the creation of a corresponding geometry?

Taken as a whole, the point-set theory is conspicuously and characteristically free from formulas. It strikes beneath the formula, which usually limits by its concreteness and definiteness. This absence of equations and formulas is an indication both of strength and of weakness. The sword cuts both ways. The point-set theory becomes less useful as the problem becomes more restricted and regular, in particular when the domain is a continuum and the problem falls under the differential calculus in which the functions are so highly civilized as to be both continuous and differentiable. At first thought there might be some surprise at the almost complete absence of point-set considerations in the differential geometry. I was much interested to find that one of my well-informed colleagues had thought of point-set theory as a tool especially adapted for use "im Kleinen"—to use a significant term of Osgood. To me, on the other hand, it had appealed because of its power "im Grossen." In its essential nature it seems to me more allied to integration than to differentiation. It is, of course, true that such concepts as limit point or point of condensation in the Lindelöf sense* apply not merely im Kleinen but im infinitesimalen Kleinen; that is to say, when the region or interval containing the point must be indefinitely contracted. The most characteristic point-set methods also involve an analysis im Kleinen. On the other hand, the greatest concepts of the theory, such as countability, derived set, perfect set, are concepts im Grossen. Thus while in the two latter we consider the limit points of the set, our attention is turned to the *totality* of such points. Similarly, the most common point-set method is one by which a property im Kleinen is first established over a dense sub-set of points and then extended to all points of the set. I doubt not that much of the characteristic strength of the point-set theory lies precisely in this union of consideration im Kleinen and im Grossen. But the latter is the dominant factor conspicuously evinced in the applications. We are, in fact, no longer content with study im Kleinen and to see in part. Modern mathematics demands, and demands with increasing insistence, methods which give control over the whole of the object under consideration, whether it be function or geometrical figure. The point-set theory complies with this demand. We have passed through a differential period in the development of mathematics. To what extent this will be succeeded by a point-set period it remains for the future to disclose.

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* A point in whose vicinity there is an uncountable number of points of the set.

I have said that the theory has had little or no application in the differential geometry of curves and surfaces. I must not, however, omit to mention one great application to geodesics upon surfaces of negative curvature made by Hadamard in a memoir* wonderful for the extensive qualitative results obtained with most sparing use of formulas. His attention is not confined in the usual manner to a single geodesic curve but he considers these curves in their totality. As in Poincaré's work on differential equations, so here success is obtained because he dares the whole. The surface of negative curvature may have any number of nappes stretching out to infinity, and may be pierced, somewhat like an anchor ring, with any number of holes. Four kinds of geodesics are distinguished: (1) closed geodesics and (2) those which are asymptotic to these, (3) geodesics which lie in finite space but are neither closed nor asymptotic to closed geodesics, and (4) geodesics which run off to infinity. If the connection of the surface is greater than 2, there is a countably infinite number of closed geodesics, and they form a set "condensed in itself," to apply a term of Hadamard which is not quite self-explanatory.† But the most interesting result relates to the set of geodesics which pass through any point of what is termed the "finite part" of the surface. The directions of those geodesics through the point which do not go to infinity form a perfect set of directions nowhere dense, the interior of the intervening sectors being filled with geodesics which run to infinity. Thus the notion of a perfect set nowhere dense has here its analogue in the set of directions at a point which belong to geodesics of the first three species.‡

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* *Liouville's Journal*, ser. 5, vol. 4 (1898), p. 27.

† The exact meaning is explained on p. 55 of his memoir.

‡ Point-set considerations of other character enter into the investigations of geodesics upon closed polyedral surfaces by Stäckel and Rodenberg; *Rendiconti del Circolo matematico di Palermo*, vol. 22 (1906), p. 141, and vol. 23, pp. 107, 111.

This work of Hadamard is the last specific example which I shall cite of the influence of point-set concepts upon geometrical theory, and in some ways it is the most remarkable one because the notion of a perfect set nowhere dense is not only carried over without modification from a set of points to a set of curves but is actually and unexpectedly realized, not through some artificially constructed example but in practical study. The example emphasizes the importance of the extension of point-set theory to sets of curves. For this a further development is necessary. Fréchet has done pioneer work in laying foundations for a theory of infinite sets of objects other than points, and I may also refer to the investigations of Arzelà, Moore, and others. The incipient geometry of infinitely many variables is also closely related and seems destined to have a very considerable importance. But what seems particularly desirable is concrete application and concrete study of sets of curves because of the broad field of application in geometry and dynamics. In this connection I should not forget to refer to Volterra's work on differentiation with respect to a curve, which involves point-set considerations to a much greater degree than does the ordinary differential calculus. But while reference to it here as an outcome of point-set theory should not be omitted, I regard it, more strictly considered, as forming a distinct branch of mathematics, lying like the differential calculus, just without or within the point-set theory according to one's point of view.

The service of the point-set theory to geometry has not by any means consisted solely in its fertilization of geometry through new ideas, but almost equally in its introduction of new methods. Time altogether fails me to enter adequately into this aspect of the subject, and it is less necessary because the methods have been in some slight degree implied in preceding considerations and are involved still more in those which follow. I should like, however, to note incidentally that the nature of a point-set entering into a problem often prescribes the method to be employed. I shall content myself with a single explicit example of one of the most typical methods, which I have already termed the condensation method. This is taken from Hilbert's* proof of the existence of geodesics upon surfaces of analytical character.

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* *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 8 (1899), p. 185. Cf. also Noble's Dissertation, Gottingen, 1901.

The length of a geodesic line upon the surface is presented under the form of an integral between the two given but arbitrary points A and B and is to be minimized by proper choice of the surface curve between the two points. The problem is therefore also one of the calculus of variations. The crucial point is to demonstrate, under the conditions given, the actual realization of the lower limit of this integral by some curve on the surface between the two points. To establish this Hilbert selects a sequence of curves between A and B whose lengths approach indefinitely this lower limit. On each of these curves the middle point is taken. A limit point of these middle points gives then a first point $P_{1/2}$ of the desired geodesic. The curves between A and $P_{1/2}$ and between $P_{1/2}$ and B are then considered in exactly the same way, and two new limit points $P_{1/4}$ and $P_{3/4}$ are obtained which are points of the final geodesic. By continuing in this manner an ordered, dense, and countable set of points $P_{n/2^m}$ is obtained whose derived set is shown to be a curve between A and B, the length of which is equal to the lower limit of the lengths of all possible curves on the surface between the two points. It is therefore the required minimum line or geodesic. By the same general method the fundamental and well-known existence theorem of Hilbert in the calculus of variations has been established.* These are but samples of geometrical problems in which the condensation method would appear to be almost the only resort. The method is easy and strong because it grips the problem beneath the formula close to the roots and seems especially adapted to the calculus of variations, in which the point-set theory has already played a considerable part.

From the calculus of variations and the study of geodesics it is not far to the applications in dynamics. Geodesics are indeed a special case of dynamical trajectories. In these applications the methods of point-set theory have, I think, exerted a greater influence than its concepts, but it is not always possible to separate them. The first noteworthy dynamical application of which I am aware was made by

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* Hilbert illustrated "the gist of his method by the example of the shortest line upon a surface," and a further development of the method was given by Noble (loc. cit.) and by Hilbert in his lectures. Important extensions are due to Bolza, Lebesgue, and Carathéodory. For a general resume see Bolza's Lectures on the Calculus of Variations, p. 245 ff.; also Carathéodory, *Math. Ann.*, vol. 62 (1906), p. 493.

Poincaré in his study of dynamical stability.* The motion of a particle or system was said by him to be stable à la Poisson when it returns an infinity of times within any prescribed distance of its initial position, however small. This is the case if the motion is periodic but can also be true when the motion is not closed and the system never returns exactly to the initial position. In celestial mechanics, the motion or transformation of space is often one which preserves volumes, thus resembling the motion of an incompressible fluid in hydrodynamics. This is presupposed by Poincaré. If the motion is continuous, we may obtain a discrete transformation by considering the change which takes place in some fixed but arbitrary time interval T. Suppose now the volume to be limited and consider the result of applying the transformation repeatedly to any initial part U_0 of the volume. Since the transformation by hypothesis preserves volume and the total volume is limited, the iteration of the transformation must bring this part into some position overlapping its original position U_0. Suppose this to occur after α_1 iterations, i. e., at the time $\alpha_1 T$. Let $U_0^{(1)}$ be the part common to the two overlapping volumes and apply the transformation in similar manner to this part. By the same argument it must come to overlap itself, say after a time interval $\alpha_2 T$. Let $U_0^{(2)}$ be the common part, and apply the transformation again to this part. Continuing in this manner, we obtain a sequence of volumes $U_0^{(n)}$ $(n = 0, 1, 2, \cdots)$, contained each in the preceding. Now by the well-known theorem of point-sets, which is the basis of the familiar method of the "Einschachtelung der Intervalle oder Bereiche," there must be at least one point common to a sequence of closed sets contained each in the preceding. If the successive pieces $U_0^{(n)}$ do not grow indefinitely small with increasing n, we may confine our attention at each successive stage of the reasoning to as small a portion of the overlapping volume as we please and thus obtain a series of volumes contracting to a single limiting point. Take this common point, and beginning de novo, apply to it the inverse transformation, thereby reversing the direction of motion of the point along its path. Since the point lies in $U_0^{(n+m)}$, it is easy to see by the reversal of the preceding considerations that it must lie in $U_0^{(n)}$ after the lapse of any time interval of the form

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* Les Méthodes nouvelles de la Mécanique céleste, vol. 3, chaps. 26 and 27; in particular, sec. 291–6.

$$- T[\alpha_{n+m} + e_{m-1}\alpha_{n+m-1} + \cdots + e_2\alpha_{n+2} + e_1\alpha_{n+1}],$$

where each $e_1, e_2, \cdots, e_{m-1}$ is either 0 or 1. As $U_0^{(n)}$ may be taken as small as we wish by choosing a sufficiently large n, it follows therefore that the point returns an infinity of times to the vicinity of its initial position, and hence its motion is stable à la Poisson.* Hence in any volume there is one, and consequently an infinite number of points whose motion is stable in the inverse transformation; also for like reason an infinite number in the direct transformation (or in both).

The question next arises whether these points of stable motion should be considered as the rule or the exception. Poincaré therefore goes on to prove that the probability of unstable motion for a particle is infinitely small. The term probability, of course, requires definition. It suffices here to say that the probability that a molecule taken at random in volume V will lie at a given instant in a given portion U of that volume is U/V. In terms of Lebesgue's measure Poincaré's conclusion can be reformulated as follows: The set of points whose trajectories are not stable has a measure 0, and accordingly the probability that a point will not return infinitely often to the vicinity of the initial position is 0.

This conclusion can be extended from particles to systems with n degrees of freedom, provided that in the corresponding n-dimensional parameter-space there is a volume or other invariant integral with positive integrand. Upon this result as a foundation rest other theorems of Poincaré which lie without my subject. Thus, though point-set considerations have entered but little into his work on celestial mechanics, they serve to establish a most pivotal result. Their introduction was, I believe, inevitable and necessitated by the very nature of the conclusion. But before discussing this, I wish to place beside Poincaré's result another illustration taken also from celestial mechanics.

One of the fundamental problems of secular variations is that of the existence of a so-called "mean motion" for the perihelion or node. Much has been writen upon the subject since the time of Lagrange, but as it now turns out, practically all subsequent deductions have been erroneous. Mathematically, the problem is as follows: Given two real series

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* Poincaré fails to contract the regions $U_0^{(n)}$ and therefore does not strictly prove the stability. "Plus exactement," he states, "il y aura des molécules qui traverseront une infinité de fois ce volume."

$$r_1 \sin(g_1 t + h_1) + r_2 \sin(g_2 t + h_2) + \cdots + r_n \sin(g_n t + h_n),$$

$$r_1 \cos(g_1 t + h_1) + r_2 \cos(g_2 t + h_2) + \cdots + r_n \cos(g_n t + h_n),$$

or the equivalent complex series,

$$\sum_{m=1}^{n} r_m e^{i(g_m t + h_m)} \qquad (r_n > 0);$$

to determine when it is possible to express their sums in the form $r \sin \omega$, $r \cos \omega$, and $re^{i\omega}$ respectively, in which the angle ω is expressible under the form $ct + X(t)$ where c is a constant and $X(t)$ is a *limited* function of t. Lagrange gave two sufficient conditions for this,* and remarked that "outside of these two cases it was very difficult and perhaps impossible to pronounce in general on the nature of the angle." Recently the problem has been completely treated for $n = 3$ by Bohl,† who shows that when the condition of Lagrange fails to hold, the determination depends upon two quantities,

$$\rho = \frac{g_2 - g_1}{g_3 - g_1} \quad \text{and} \quad \zeta = \frac{1}{\pi}(\omega_3 + \rho\omega_2),$$

in which ω_1, ω_2, ω_3 denote the angles of a triangle with sides r_1, r_2, r_3. Now from Lagrange's work it was known that the mean motion exists when ρ is rational. Bohl shows further that for every pair of values (ρ, ζ), the ratio ω/t has for $t = +\infty$ a limit c, but that, on the other hand, there is a set of values everywhere dense in the region $|\rho| < 1$, $|\zeta| < 1$, for which the remainder $X(t)$ is not limited and for which accordingly there is no mean motion. Since also the points (ρ, ζ) with commensurable ρ are everywhere dense, Bohl concludes that it will be impossible ever to ascertain by experimental data whether or not the mean motion exists. So far as I have noticed, Bohl does not actually use point-set theory in his proof, though sets of number-points frequently appear. More recently F. Bernstein‡ has shown by point-set methods that the total set of points (ρ, ζ) for which the mean motion exists has a Lebesgue measure 0. Bernstein then advances a philosophical or physical principle in which he asserts that in physical measurement we can never take account of a set of points of measure 0; consequently, for us there can be no mean motion except in the Lagrangian cases.

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* Namely, that some one of the r_i should be greater than the sum of all the others, or $n = 2$.

† *Crelle's Journal*, vol. 135 (1909), p. 189.

‡ *Math. Ann.*, vol. 71 (1911–12), p. 417.

Into a discussion of this interesting but questionable principle I shall not enter. What I do wish to do, however, is to point out that both this result and that of Poincaré are without doubt typical of the exceeding complications of physical nature which must increasingly engage the mathematician's attention. Thus, for example, orbits possessing a specified character and those which do not, may be tangled densely together in the space of the parameters on which these orbits depend. It is to the point-set theory that one must turn to control the two sets of parameters both quantitatively and qualitatively. Already groups of exceptional points of measure 0 have been discovered with considerable frequence in the theory of functions. By allowing now for such a set of exceptional points in an interval or field we can often obtain the necessary or sufficient condition for a desired result, for instance in the case of Fourier's series. Prior to the introduction of the Lebesgue measure no general theorems were visible because when the cases pro and the cases con were both infinite in number, there was no means of overcoming the difficulty and reaching a definite result. Undoubtedly, also, we must expect to control many phenomena in astronomy and dynamics only by admitting a set of exceptions of measure 0.

The introduction of measure also broadens the scope of the theory of probability, which is so fundamental to a large part of physical speculation. For by its use account can be taken of infinite sets of discrete points, dense and complicated; and more generally, exceptional cases, infinite in number, can be brought under the theory of probability by attributing to them a probability proportional to the measure of the corresponding points in the space of the parameters upon which they depend, though care must be taken inasmuch as measure is not an invariant of analysis situs and hence may be dependent on the parameters introduced. This application of measure is as yet prospective rather than actual. In this connection I should mention also Borel's treatment of probability for a countably infinite number of cases.

Two other aspects of the above discussed work of Poincaré seem to me to be also worthy of attention. In the first place, to get a grip upon dynamical stability Poincaré abandons the infinitesimal transformation and considers instead the result of iterating a finite transformation arising in time T. From a single point or an arc traversed in time T there arises by iteration of the transformation a whole set of points or arcs, which are capable of point-set treatment. A wide field is opened in this manner for the future application of point-set methods, especially in dynamics and the theory of geometric transformations. As an instance of application to the latter subject I may cite my determination* of the form of a one-parameter group of projective transformations when a single finite substitution is given instead of the generating infinitesimal substitution of Lie. I need not plead the prospective usefulness and importance of this iterative point-set method with those of you who heard Professor Birkhoff's paper at our last summer meeting. But I should also add that in his memoir on the movement of dynamical systems† he has obtained some notable results on dynamical stability by what is essentially direct point-set treatment of the infinitesimal transformation.

The remaining aspect of Poincaré's transformation which I wish to recall is its assumed property of conserving volume. Later, in his last published memoir,‡ Poincaré put before the mathematical world an unproved theorem based on a similar restriction. If, namely, a circular ring is changed into itself by a continuous 1–1 transformation *with preservation of areas* and in such a way that the points of the one boundary are moved clockwise and those of the other boundary anti-clockwise, there must be at least two invariant points within the ring. Clearly, this is a problem of analysis situs, but with an accessory condition of preservation of areas which makes it at the same time a problem of hydrodynamics. Professor Birkhoff's§ brilliant solution is, I hope, known to you all. His method contains a highly ingenious but, as he himself would recognize, somewhat artificial device. The natural attack seems to me to be through point-set methods, as was apparently the thought of Poincaré. Results of similar import in analysis situs, but without the restriction of the preservation of volume, have been attained through point-set considerations by Brouwer, who shows, for example, that every continuous 1–1 transformation of a sphere of even dimensions into itself without reversal of circuit sense, or of a sphere of odd dimensions with such reversal, leaves at least one point invariant. It may be confidently expected that many results in hydrodynamics will be obtained when the requirement of invariability of area or volume is added.

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* *Transactions*, vol. 13 (1912), p. 353.

† *Bulletin de la Société Mathématique de France*, vol. 40 (1912), p. 305.

‡ *Rendiconti del Circolo matematico di Palermo*, vol. 33 (1912), p. 375.

§ *Transactions*, vol. 14 (1913), p. 14.

I turn now from celestial mechanics and hydrodynamics to the field of statistical mechanics. Consider a so-called gas-model, composed of a very large number N of molecules having each r degrees of freedom. The corresponding canonical equations of Hamilton,

$$\frac{dq_s^{(k)}}{dt} = \frac{\partial E}{\partial p_s^{(k)}}, \qquad \frac{dp_s^{(k)}}{dt} = -\frac{\partial E}{\partial q_s^{(k)}} \qquad \begin{matrix} (s = 1, \cdots, r), \\ (k = 1, \cdots, N), \end{matrix}$$

involve r generalized coordinates $q_s^{(k)}$ and r moments $p_s^{(k)}$ and the total energy E. The $2rN$ quantities $q_s^{(k)}$, $p_s^{(k)}$ determine a "phase" of the system. For geometrical representation a $2rN$-dimensional space is required, called the phase space (Phasenraum). Now it was a great discovery of Liouville, made independently a second time by Boltzmann and Poincaré, that in consequence of Hamilton's equations volume must be an invariant integral of motion in this $2rN$-dimensional space. The given value of the energy E determines what is called the energy-surface within this space, upon which the molecules move in a one-dimensional path. As the foundation of their gas and statistical investigations Boltzmann and Maxwell assumed that the system was "ergodic." By an ergodic system is to be understood one in which a molecule so moves as to pass through every single point of the energy surface upon which it lies. Since, furthermore, there is a unique motion at any point of the phase-space, all the particles upon the energy-surface are describing one after another exactly the same curve. Now the possibility of such motion was assumed but not proved by Boltzmann. Mathematically considered, this is tantamount to the assumption of the existence of a space-filling curve *without double points*. For if there were anywhere a double point in the path of a particle in the phase-space, the corresponding gas-model would pass twice through the same state, and the particle would in consequence begin to repeat its motion without describing the rest of the curve. What would have seemed more improbable than that the freak space-filling curves first invented by Peano could ever have a practical bearing! Imagine the scorn of the average physicist for such impractical, crazy, mathematical pathology. Yet here is a master physicist Boltzmann not merely postulating the existence of space-filling curves upon his energy-surface but making an impossible assumption, and therefore an actually crazy one, that such curves can exist[*] without double points. The proof of their non-existence belongs to the point-set theory and results from Brouwer's theorem[†] that it is impossible to build two regions of different dimensions—in the present case[‡] the $(2rN - 1)$-dimensional space-surface and the one-dimensional time-axis—one upon the other by means of a continuous 1–1 transformation.

The impossibility of an ergodic system has been considered also from another standpoint by Plancherel,[§] who shows that the path of a particle has a surface measure 0 and hence cannot fill the energy-surface as desired. It follows immediately that the number of different path-curves on the surface must be more than countable.[||] I may mention further that Rosenthal[¶] has replaced Boltzmann's supposition by a quasi-ergodic hypothesis, according to which each path on the energy-surface comes within any prescribed distance of every point of the surface.

In essence the statistical mechanics is a universal application of averages or, in other terms, of a theory of probability. It is therefore to be anticipated that numerous other applications will be here made of the theory of point-sets.

The examples which I have brought to your attention are, I trust, sufficient to convince you that the point-set theory has vital relations to the study of dynamical trajectories as well as to geometry. I have preferred to illustrate its rôle with concrete examples rather than to indulge in abstract discussion. In so doing we have dug down to the very roots of geometry and dynamics but we have also touched the branches where the shoots are pushing out. Too much must not be claimed. Yet if the point-set theory is not the sap of the tree, it is at least one of its most indispensable ingredients. The usefulness of any mathematical theory must be determined not by its isolation but by its ability to combine with other theories. In this the point-set theory has shown itself most elastic. Had I more time I would attempt to show you that it offers many of the same advantages as analytical geometry. While strong for analysis and decomposition, it is equally strong on the constructive side. Complex groupings of points are made simple, and the way is thus prepared for new discovery. And above all, in its development the arithmetization of analysis is kept close to geometrical intuition.

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[*] Cf. Rosenthal, *Annalen der Physik*, vol. 42 (1913), p. 796.

[†] *Math. Ann.*, vol. 70 (1911), p. 161.

[‡] For this case the simple proof given by Jürgens suffices; cf. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 7 (1898), p. 54.

[§] *Annalen der Physik*, vol. 42, p. 1061.

[||] This is also shown by Rosenthal (l. c.) who employs for that purpose Baire's concept of point-sets of the first and second categories.

[¶] *Annalen der Physik*, vol. 43 (1914), p. 894.

In tracing the service of the theory of point-sets in geometry and dynamics, we have found only in part achievement, in part present evolution and promise. But it is precisely because of this mocking incompleteness that I have chosen for my topic today the rôle of the point-set theory in geometry and dynamics, trusting that for you also this will be its lure.

University of Wisconsin,
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AN ENUMERATION OF INTEGRAL ALGEBRAIC POLYNOMIALS.

BY PROFESSOR A. B. FRIZELL.

(Read before the American Mathematical Society, January 1, 1915.)

The proof given by Weber* that the algebraic numbers form a countable set orders them according to the values of a certain function of the coefficients in their defining equations. The present note suggests a more direct enumeration of these equations. The algebraic polynomials $\sum_{i=0}^{n} a_i x^{n-i}$ in which all coefficients are natural numbers can be put into one-to-one correspondence with the set of natural numbers by the following device. To every kx^r assign the number p_r^k, where $p_0 = 2$ and p_r denotes the rth odd prime when $r \neq 0$. To the sum $k_1x^{r_1} + k_2x^{r_2}$ assign the product $p_{r_1}^{k_1} \cdot p_{r_2}^{k_2}$, and so on. Thus the polynomial $x^8 + 3x^4 + 2$ will have the number $p_8 \cdot p_4^3 \cdot p_0^2 = 23 \times 11^3 \times 2^2 = 122{,}452$. Conversely, the polynomial numbered $360 = 5 \times 3^2 \times 2^3 = p_2 \cdot p_1^2 \cdot p_0^3$ will be $x^2 + 2x + 3$.

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* Algebra, Bd. II, p. 824.

The polynomials in which negative integers occur as coefficients may be put into one-to-one correspondence with the set of positive fractions in their lowest terms by substituting division for multiplication in the above process. Thus to the polynomial $x^8 - 3x^4 + 2$ will be assigned the fraction

$$\frac{p_8 \cdot p_0^2}{p_4^3} = \frac{92}{1331}.$$

These fractions can be counted off diagonally from the well-known rectangular array where every element in the nth row has the denominator $n + 1$, and numbered with odd numbers, reserving the even numbers for the polynomials with positive coefficients, so that finally $x^8 + 3x^4 + 2$ is to be numbered 244,904. For example, to identify number $605 = 2 \times 303 - 1$, we first obtain the greatest value of n which makes

$$\frac{n(n+1)}{2} \leqq 303.$$

This shows that number 303 in the rectangular array is the third fraction in the 25th diagonal, counting upwards. To avoid repetitions we will agree that the highest term in every polynomial shall be positive. This requires that the numerator of each fraction shall contain a prime factor greater than any in the denominator. Hence number 303 is

$$\frac{11}{24} = \frac{p_4}{p_1 \cdot p_0^3} \sim x^4 - x - 3.$$

To determine the number which belongs to $x^8 - 3x^4 + 2$, we count in row 1,330 up to 92, which is the 41st. Therefore 92/1331 is in diagonal 1,370. The sum of the first 1,369 diagonals is 937,765 and the required number is 1,875,611.

McPherson, Kans.

MR. PAASWELL'S APPEAL TO PRODUCING MATHEMATICIANS.

BY PROFESSOR C. N. HASKINS.

It is always a service to the scientific world to set before it a definite statement of some of its important unsolved problems. Such a service Mr. Paaswell has rendered,* in calling attention to certain outstanding mathematical problems of engineering. His thesis is, essentially, that the available mathematical analysis of these problems is, (*a*) exact, but impracticably complex; or, (*b*) practicably simple, but inexact; or, (*c*) non-existent, save for empirical formulas. He therefore urges producing mathematicians to turn their attention to bettering this state of affairs.

The question naturally arises, "Why does not the engineer himself supply the needed remedy?" Mr. Paaswell suggests several answers, viz.,

(1) The mathematical curricula of the schools of applied science are not sufficiently intensive or extensive.

(2) Modern mathematical treatises and dissertations are so written as to be incomprehensible to those not already familiar with the subject.†

(3) "Hardly any treatise has attempted to discuss or analyze the serious problems of the applied science professions."

Of these answers (3) is in effect merely a restatement of the thesis. The suggestion (1) deserves careful consideration. The required mathematical courses of the better schools of engineering represent very nearly the maximum, both in depth and extent, of what can be effectively assimilated and used by the average student in such schools. An increase of significant amount in these required courses would result either in depriving the engineering profession of many useful and successful members, or in lowering the general standard of attainment demanded in those courses. Unquestionably the student engineers of real mathematical aptitude would benefit by the suggested increase. On the other hand a large number of students who have only moderate mathematical ability, but who nevertheless have compensating qualities which will permit them to develop into useful if not great engineers, would be actually injured by a required course of extent and depth seriously exceeding their powers. Furthermore the mathematically gifted students would not, unless some system of segregation were effected, receive the full benefit of the increased course, since they would be retarded by their less gifted fellows. An increase in the *required* courses, therefore, appears to be at best an inefficient way of leading to the desired result. Much, however, may be expected of more numerous, broader, and deeper elective courses if a suitable type of student can be encouraged to avail himself of them.

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* BULLETIN, vol. 21 (1914), p. 127.

† Mr. Paaswell also contends that "the tone of modern works is not that of disseminating new ideas, but rather that of clothing ideas already familiar to readers in slightly different form." This position seems hardly tenable. Witness, for example, the monographs of the Borel series.

Suggestion (2), however, expresses the real difficulty, and possibly contains implicitly the key to the solution.

The problems mentioned by Mr. Paaswell belong both to engineering and to mathematics. Now engineering and mathematics are two autonomous but friendly professions. The engineer is by virtue of his profession no more a mathematician than is the mathematician an engineer. The engineer's despair of keeping up with the trend of modern mathematical thought has its exact counterpart in the mathematician's despair of keeping up with modern engineering thought and practice. To attack the problems under discussion, however, the despairing engineer needs must add mathematics to his engineering, or the despairing mathematician engineering to his mathematics. Mr. Paaswell suggests the latter alternative, the mathematician may well suggest the former.

The situation described by Mr. Paaswell is, however, not unique in character. In the field of public health there exists an analogous state of affairs, and in that field remedial measures are now being taken which bid fair to lead to a successful result,* and from which we may gather ideas applicable to the present situation.

It has been recognized that, while both physicians and engineers have developed themselves into successful health officers, yet neither the physician nor the engineer is by reason of his own professional training alone wholly competent to attack with success the problems of the control of public health. The demand for officers fitted to cope with these problems has led certain institutions to establish distinct courses to provide both the medical and the engineering training which such officers require. But it is not expected that these courses should be a part of the training of every physician and of every engineer, nor even that every school of medicine or of engineering should establish such courses.

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* Cf. Whipple, G. C., *Science*, n. s., vol. 40 (1914), p. 581.

There already exist under university or government control several stations or institutions devoted to research in the problems of engineering. This research is naturally largely experimental, and these stations provide the opportunity of a career for the man who has the aptitude and the desire for this work. If such stations enter upon the field of the mathematical problems of engineering the door will be opened to a career in this line also.

With a definite demand for men competent to attack the mathematical problems of engineering will come the inducement for men to train themselves for the work, and with this inducement, a demand for suitable courses. To meet this demand a few institutions, already strong in both mathematics and engineering, may well organize graduate courses analogous to those which now lead to the degree of Doctor of Public Health.

FUNCTIONS OF LINES.

Leçons sur les Fonctions des Lignes. Par VITO VOLTERRA. Recueillies et rédigées par JOSEPH PÉRÈS. Paris, Gauthier-Villars, 1913. 8vo. vi+230 pp.

THESE lectures were delivered by Volterra at the Sorbonne during the months from January to March, 1912, and were later published as one of the Borel series of monographs on the theory of functions. It would be difficult to determine precisely the historical origin of functions of lines. Special cases of such functions, for example the ordinary definite integral or the integrals of the calculus of variations, have occupied a large share of the attention of mathematicians since the beginnings of the calculus itself. But the conscious formulation of the definition of a function of a line and its derivative, and the study of a general theory, belong to a recent period of investigation in which Volterra has been an earliest pioneer. The development of our knowledge of functions of lines and their applications, since Volterra's first published papers on the subject in 1887, has been due largely to his enthusiasm and to his confidence in the future importance of the theory.

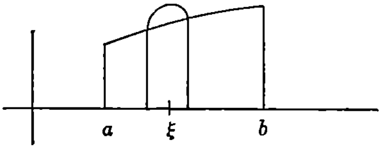
A function of a line may be conceived as a generalization from a function $F(y_1, y_2, \cdots, y_n)$ of a finite number of variables to a function

$$F|[f_a^b(x)]| \tag{1}$$

depending upon the infinity of y-values belonging to a curve $y = f(x)$ over an interval $a \leqq x \leqq b$. This process of passing from a conception involving a finite number to a conception involving an infinite number of mathematical symbols or operations is one of the most striking features of the mathematical thought of the present time. It is a process whose numerous possibilities of application Volterra repeatedly emphasizes in his Leçons, and which has been a favorite instrument in the prosecution of his varied researches.

Perhaps the most familiar example of such a process is the definite integral, the limit of a sum of a finite number of terms as the number of terms is indefinitely increased and the magnitude of each of the terms suitably diminished. Many centuries elapsed while this notion was haltingly developed into the basis of the modern integral calculus, and two more were required before the full importance was realized of the generalization in a similar way of other processes than sums. Then within a period of twenty-five years came the theories of functions of lines, of linear integral equations as generalizations from a finite to an infinite number of linear equations, of integro-differential equations as limiting conceptions corresponding to finite systems of simultaneous partial differential equations, and of other generalizations of the same sort with which the reader is doubtless familiar. It is with the three conceptions just mentioned specifically that the Leçons of Volterra are primarily concerned.

In an introductory chapter of great interest he has sketched the historical development of the fundamental principles of the integral calculus, and has shown how inevitably the tendencies of modern analysis and mechanics, but especially the mechanics of heredity, have led to the study of functions of lines. The theory of these functions is as yet in its infancy, and no final classification or complete discussion of its various branches can at present be made. But Volterra indicates briefly at this point the directions of the researches which have hitherto been made, and some of the questions which must be answered in the future.



In Chapters II, III, and IV the fundamental properties of functions of lines are developed. Let ΔF be the difference between the value of the function (1) taken along the curve $y = f(x) + \varphi(x)$ with the hump shown in the figure, and the value of F along the curve $y = f(x)$ without the hump; and let σ be the area of the hump. Then Volterra defines the derivative of F at the value ξ to be the limit

$$F'|[f_a^b(x), \xi]| = \lim_{h=0,\ \epsilon=0} \frac{\Delta F}{\sigma}, \tag{2}$$

where h is the length of the interval over which $\varphi(x)$ is different from zero, and ϵ is the maximum of the absolute value of φ. If F satisfies suitable restrictions this derivative will exist, and for a family of variations $y = f(x, \alpha)$ containing the curve $y = f(x)$ for $\alpha = 0$, the formula

$$\delta F = \int_a^b F'|[f(x); \xi]|\delta f(\xi, 0)d\xi,$$

so important in the calculus of variations, can be proved. The differentials in this formula are taken with respect to α. If F has also higher derivatives of all orders

$$F^{(n)}|[f_a^b(x), \xi_1, \xi_2, \cdots, \xi_n]|,$$

whose definitions will be readily inferred, and satisfies other suitable conditions, it will have an expansion of the form

$$\begin{aligned} F|[f(x) + \psi(x)]| &= F|[f(x)]| + \int_a^b F'|[f(x), \xi_1]|\psi(\xi_1)d\xi_1 \\ &+ \frac{1}{2}\int_a^b \int_a^b F''|[f(x), \xi_1, \xi_2]|\psi(\xi_1)\psi(\xi_2)d\xi_1 d\xi_2 + \cdots, \end{aligned}$$

a generalization of Taylor's formula.

It is not always true that two arbitrarily chosen functions of x and y will be the first partial derivatives of a third such function. The derivative (2) is really the partial derivative of F with respect to only one of the infinitely many variables which it involves, namely the value of y corresponding to $x = \xi$. It is not to be expected, therefore, that an arbitrarily chosen function (2) will always be the derivative of a function (1). Volterra deduces the conditions under which a function of the form (2) will be the derivative of a line function, and shows how the anti-derivative may be calculated with the aid of a generalization of Stokes' theorem.

If a curve $y = f(x)$ minimizes a line function F, the derivative (2) of F must vanish identically along the curve. For the line functions of the calculus of variations this condition leads to the usual ordinary differential equations due to Euler, but, as Volterra shows, it is easy to give an example for which the equation so found is of the type of a linear integral equation, or an integro-differential equation. The state of the theory at present does not justify a complete classification of all of the types which may arise.

Another interesting type of equations is met with when one attempts to generalize for double integrals the Jacobi-Hamilton theory associated with an integral of the form

$$F = \int_{x_0}^{x} V(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx.$$

The minimizing curves for such an integral are a family containing $2n$ parameters, and the parameters may in general be determined so that the resulting curve passes through two arbitrarily selected points, $(x_0, y_1^0, \cdots, y_n^0)$ and $(x, y_1, \cdots, y_n)$. The integral F is then a function of the coordinates of these points, called the extremal integral. It satisfies a partial differential equation of the form

$$\frac{\partial F}{\partial x} = H\left(x, y_1, \cdots, y_n, \frac{\partial F}{\partial y_1}, \cdots, \frac{\partial F}{\partial y_n}\right). \tag{3}$$

For a double integral

$$\int\int V\left(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) dxdy$$

the minimizing surfaces are solutions $z = f(x, y)$ of a partial differential equation of the second order, and are determined by initial values $f_0(s)$ assigned arbitrarily to f on a closed contour L with length of arc denoted by s. The extremal integral is then a function $F|[f_0, L]|$ of the contour L and the values f_0, and the equation corresponding to (3) has the form

$$F'_L|[f_0, L, s]| = \frac{1}{2}\left(\frac{\partial f}{\partial s}\right)^2 - \frac{1}{2}F'^2_{f_0}|[f_0, L, s]|.$$

Equations of this type have been studied quite recently by Hadamard and Lévy, and are called by Volterra "équations aux dérivées fonctionelles."

The problem of determining the minimizing curves for a function of a line suggests the generalizations of the theory of implicit functions which are studied in Chapter IV. The equations

$$F_i(f_1, f_2, \cdots, f_n; \varphi_1, \varphi_2, \cdots, \varphi_m) = 0 \quad (i = 1, 2, \cdots, n),$$

which determine the variables f in terms of the φ's, correspond to a single equation of the form

$$F|[f_a^b(x), \varphi_{a'}^{b'}(x), \xi]| = 0 \tag{4}$$

when indices ranging over discrete integers are replaced by the index x ranging over the two continua $a \leqq x \leqq b$, $a' \leqq x \leqq b'$. The forms of this equation studied in detail by Volterra are, first, one corresponding to the linear equations which give rise to the theory of linear integral equations with variable upper limits; second, the case when the function F in (4) is the sum of two parts, one involving f and the other φ, and each part expansible by the generalization of Taylor's formula described above; and, finally, an equation (4) whose first differential has a special form. There remain, apparently, many cases yet to be studied, but the methods used by the author and the results which he has found are most interesting and suggestive.

In Chapter V Volterra begins his study of integro-differential equations with a special case. The equation considered is

$$\Delta^2 u(t) + \int_0^t \left[\frac{\partial^2 u(\tau)}{\partial x^2} f(t, \tau) + \frac{\partial^2 u(\tau)}{\partial y^2} \varphi(t, \tau) + \frac{\partial^2 u(\tau)}{\partial z^2} \psi(t, \tau) \right] d\tau = 0, \tag{5}$$

where

$$\Delta^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

and $u(x, y, z, t)$ is to be determined, an equation which he designates as the simplest integro-differential equation and the most important from the point of view of physics. The equation may be regarded as a generalization from a finite system

$$\Delta^2 u_i + \sum_{j=1}^{i-1} \left[a_{ij} \frac{\partial^2 u_j}{\partial x^2} + b_{ij} \frac{\partial^2 u_j}{\partial y^2} + c_{ij} \frac{\partial^2 u_j}{\partial z^2} \right] = 0 \quad (i = 1, 2, \cdots, n)$$

when the indices i and j are replaced by continuous variables t and τ, and the sum by the corresponding definite integral. The methods of treating this equation and the results found are analogous to those of the potential theory. There is an equation adjoint to (5) whose solutions satisfy a law of reciprocity with those of (5). There is a fundamental solution corresponding to the quotient $1/r$ in the potential theory, which Volterra determines with the help of the process of passing from a finite to an infinite system, so often mentioned above. Finally, formulas analogous to those of Green are deduced, expressing the values of a solution for $t = \theta$ at an arbitrarily selected point in the interior of an xyz-domain, in terms of its values for $0 \leqq t \leqq \theta$ on the boundary of the domain.

Chapters VI–VIII are devoted to the theory of elasticity and to concepts concerning functions of lines which have applications in this theory. The displacements in the interior of an elastic body are determined, in the classical theory of elasticity, as solutions of a system of partial differential equations, with initial conditions on the bounding surface defined by tensions or displacements there prescribed. This is for the case when displacements at a given time are supposed to depend only upon the tensions which exist at the same instant. For the hereditary theory of elasticity, when the displacements at a time t are supposed to depend upon the tensions at all times preceding t, Volterra shows that the former may be expressed in terms of the latter by means of line functions, and that the differential equations for the displacements then become integro-differential equations. For the case when the body is isotropic and the line functions mentioned above linear, an assumption analogous to the customary neglect of terms of an analytic function of degree higher than the first, Volterra sketches the theory with some detail. Theoretically the elastic problem so formulated involving heredity is completely solvable when displacements on the contour are given. The steps in the theory are analogous to those outlined for the integro-differential equation of Chapter V.

In Chapter VII a principle which Volterra calls "la condition du cycle fermé" is developed at length, and the importance of its applications in the theory of heredity is emphasized. If the condition of the closed cycle is satisfied in a problem involving heredity, the so-called coefficients of heredity will be functions permutable with unity in the sense described in Chapter IX, where the theory of such functions is systematically treated. The remainder of Chapter VII is devoted to a discussion of heredity in electromagnetic theory, and an application in this connection of the integro-differential equation of Chapter V.

In Chapter VIII a solution is given of the problem of the isotropic elastic sphere under the influence of heredity, when displacements on the contour are prescribed. Many of the principles involved in this special though very important problem are illustrations of the more general developments of later chapters.

The problem of the isotropic elastic sphere under the influence of heredity presents much greater difficulty when the tensions instead of the displacements on the surface of the sphere are the data given in advance. In order to attain a solution Volterra introduces a theory called the composition and permutability of functions, and he is led thereby to results which far surpass in generality the needs of this special problem. There are two kinds of composition, that of the first kind with its applications being developed in Chapters IX–XI. For two functions F_1 and F_2 the result of a composition of the first kind is a new function

$$\overset{*}{F}_1\overset{*}{F}_2(x, y) = \int_x^y F_1(x, \xi)F_2(\xi, y)d\xi.$$

The two functions are said to be permutable if

$$\overset{*}{F}_1\overset{*}{F}_2 = \overset{*}{F}_2\overset{*}{F}_1.$$

For a given set of functions the operation of composition applied to elements of the set is always associative and, if every pair is permutable, commutative. If the set so found is extended to contain all linear combinations of its functions of the form

$$c_0 + c_1F_1 + \cdots + c_nF_n$$

the properties of associativity and permutability of composition in the set are still preserved, provided that composition for two functions $c_1 + F_1$, $c_2 + F_2$ is defined by the equation

$$(c_1 + F_1)(c_2 + F_2) = c_1c_2 + c_2F_1 + c_1F_2 + \overset{*}{F}_1\overset{*}{F}_2.$$

But the set permits also a further important extension without disturbing these properties. For if in a convergent series the variables $z_1, z_2, \cdots, z_n$ are replaced by functions $F_1, F_2, \cdots, F_n$ of the set, and multiplications replaced by compositions, the new series will be convergent provided only that the moduli of $F_1, F_2, \cdots, F_n$ are finite. Furthermore the series function will be permutable with the ones previously defined.

By means of these sets of permutable functions, whose compositions obey the laws of algebra described above, a large variety of integral and integro-differential equations may be solved. For if the equation $F(z_1, z_2, \cdots, z_n) = 0$, where F is analytic at the origin, has a solution $z_n = f(z_1, z_2, \cdots, z_{n-1})$, the corresponding integral equation

$$F(\overset{*}{F}_1, \overset{*}{F}_2, \cdots, \overset{*}{F}_n) = 0,$$

formed by replacing the variables z by permutable functions as described above, has the solution

$$F_n = f(\overset{*}{F}_1, \overset{*}{F}_2, \cdots, \overset{*}{F}_{n-1}).$$

Furthermore an algebraic differential equation

$$\Phi\left(z_1, z_2, \cdots, z_n, F, \frac{\partial F}{\partial z_1}, \frac{\partial F}{\partial z_2}, \cdots\right) = 0$$

with a solution $F(z_1, z_2, \cdots, z_n)$ analytic at the origin, takes the form

$$\Psi\left(z_1, \cdots, z_n \mid \xi_0, \xi_1, \cdots, \xi_n \mid f, \frac{\partial f}{\partial z_1}, \cdots\right) = 0$$

when z_i is replaced by $\xi_i z_i$ $(i = 1, 2, \cdots, n)$, and F by f/ξ_0. If $\xi_0, \xi_1, \cdots, \xi_n$ are in turn replaced by permutable functions, and multiplications involving these quantities or f and its derivatives by compositions, then the last equation becomes an integro-differential equation

$$\psi\left(z_1, \cdots, z_n \mid \overset{*}{F}_0, \cdots, \overset{*}{F}_n \mid \overset{*}{f}, \frac{\partial \overset{*}{f}}{\partial z_1}, \cdots\right) = 0$$

whose solution is

$$f = \overset{*}{F}_0 \overset{*}{F}(z_1 \overset{*}{F}_1, \cdots, z_n \overset{*}{F}_n).$$

The function so defined has the remarkable property that it is convergent for all values of the variables z. Volterra shows in particular that the solutions of the integral or integro-differential equations considered in the preceding chapters can be reduced in this way to the solutions of ordinary algebraic or differential equations, the solutions of the latter being in many cases already known.

An interesting example is the equation

$$\frac{dU}{dz} = U + 1$$

with the known solution

$$U = z + \frac{z^2}{2!} + \cdots.$$

After substitutions of ξz for z, and V/ξ_0 for U, the differential equation becomes

$$\frac{dV}{dz} = \xi V + \xi\xi_0.$$

If ξ_0 is replaced by unity, and ξ by a function F, as agreed above, the new integral equation and its solution are

$$\frac{dV(z \mid x, y)}{dz} = F(x, y) + \int_x^y F(x, \xi) V(z, \xi, \eta) d\xi,$$

$$V(z \mid x, y) = zF(x, y) + \frac{z^2}{2!} \overset{*}{F}{}^2(x, y) + \cdots.$$

Further the addition formula

$$U(z + u) = U(z) + U(u) + U(z)U(u)$$

gives rise to an integral addition formula

$$V(z + u \mid x, y) = V(z \mid x, y) + V(u \mid x, y) + \int_x^y V(z \mid x, \xi) V(u \mid \xi, y) d\xi$$

for the solution of the integral equation. The integral transcendental function $V(z \mid x, y)$ so defined plays an important rôle in the solution of the problem of the elastic sphere with displacements on the surface given. The solution of the problem when tensions are given may also be effected with the help of the theory of permutable functions

The above results form the content of Chapters IX and X. In Chapter XI further questions concerning permutable functions of the first kind are discussed. Among them may be mentioned a definition of order of a function, the determination of all functions permutable with a given function of a given order, and the application of these results to the solution of integral equations at critical points.

In Chapters XII and XIII composition and permutability of the second kind are defined and applied to the solution of integral and integro-differential equations with fixed limits. The result of a composition of the second kind on two functions F_1 and F_2 is defined by the equation

$$\overset{**}{F}_1\overset{**}{F}_2(x, y) = \int_0^1 F_1(x, \xi) F_2(\xi, y) d\xi.$$

While the fundamental properties of such compositions are similar in many respects to those of composition of the first kind, there are still important differences, as one would expect from the analogy with the theories of integral equations with variable or fixed limits. It will perhaps be sufficient here to say that the underlying idea is again to pass from algebraic or differential equations to integral or integro-differential equations, but this time with limits fixed instead of variable. The solutions of the former types of problems will go into those of the latter by the substitution of compositions for multiplications.

The final chapter of the Leçons is devoted to a historical summary of the more recent developments in mathematical theories of mechanics, and a discussion of the place which the theory of heredity should take in this domain. Some writers have questioned the importance of discussions of heredity on the ground that the behavior of an elastic body, for example, is completely determined after a time t_0 by its state at $t = t_0$. According to this view the discrepancies between theory and observation are due to inadequate experimental facilities for the determination of interior initial conditions. Volterra, on the other hand, contends that just as bodies react upon each other at a distance, so it is possible that tensions and displacements separated by an interval of time may be related. Even if the contention of those who question the philosophical basis of the theory is granted, nevertheless the agreement between experimental results and the elastic theory involving heredity is in itself a justification. It seems clear to Volterra that, in view of the difficulty of determining initial conditions experimentally, and in the absence of other theories agreeing with experiment, the theory of heredity offers the only explanation at present possible of a large class of phenomena. He cites the experiments of Webster and Porter in the theory of sound as a cogent illustration, and his own success in the mathematical solution of problems in the theory of elasticity with heredity must be regarded as a potent argument in favor of his point of view.

G. A. BLISS.

SHORTER NOTICES.

List of Prime Numbers from 1 *to* 10,006,721. By D. N. LEHMER. Washington, D. C., Carnegie Institution of Washington, 1914. xvi+133 pp.

THERE are several reasons why number theorists will welcome most heartily the publication of this volume.

First, it answers with utmost directness the question, arising at almost every stage of number theoretic computation, whether or not a proposed number (under ten millions) is prime. Here the question of absolute accuracy of a table is paramount; the user of such a table has no practical means of checking the accuracy of an entry and if he relies upon an erroneous entry his conclusions will be wholly wrong. It is thus quite different from the case of ordinary tables (those of the values of a continuous function), since it is there only a question of approximation and a grossly erroneous error should be detected by the user of the table. The present table probably possesses the same high degree of accuracy as Lehmer's Factor Table for the First Ten Millions, published by the Carnegie Institution in 1909,—the accuracy of which was emphasized in the writer's review in the BULLETIN, volume 17 (1910–11), pages 36–38. In fact, the present List of Primes was compiled from that Factor Table with ample safeguards against the introduction of new errors. It is true that the Factor Table tells whether a proposed number under ten millions is prime or not; but the necessity of condensation required the use there of a scheme of entry which demands a little extra time and trouble in locating a proposed number. Hence the primality of a number is much more easily detected by the use of the new table.

In the second place, the present table affords information, more accurate than hitherto obtained, concerning the distribution of primes. The table is so arranged that it tells at once (and without the loss of space) the rank of a given prime, for example, that 17 is the 8th prime. Hence the table tells directly the number of primes less than any proposed limit not exceeding ten millions. In his Introduction the author has tabulated and compared the actual count of primes with the values given by the formulas of Riemann, Tchebycheff, and Legendre, at intervals of fifty thousand up to ten millions. Moreover, he has given an account in simple language of the history of those empirical formulas for the number of primes less than a given limit. Finally, there is a description of the manuscript factor table of Kulik up to a hundred millions.

In the third place, the publication of this convenient and compact table of 133 pages (though of large dimensions) of the primes below ten millions has made it practicable that every worker in the theory of numbers shall have in his kit this most essential tool.

The printing has been done very clearly on paper selected after numerous experiments so as to withstand the ravages of time. And this is appropriate for such a monumental work. Completed after ten years of arduous work on the part of Lehmer and assistants provided by the Carnegie Institution, and printed and published by that Institution, the Factor Table and the List of Primes form truly a great monument both to Lehmer and to the Carnegie Institution.

L. E. DICKSON.

The Pell Equation. A history of the equation $x^2 - Ay^2 = 1$, table of solutions from $A = 1{,}501$ to $A = 1{,}700$, bibliography, table of continued fractions for $\sqrt{A}$. By EDWARD EVERETT WHITFORD. Lancaster, Pa., The New Era Printing Company, 1912. iv+193 pp.

THIS book presents in a very readable form an historical account of this famous problem. The author first discusses, in considerable detail, various efforts of the ancient Greek and Hindu mathematicians to find rational approximations to square roots, showing how these ultimately depend upon solutions of special cases of the Pell equation.

While these efforts and those of later mathematicians to find square root approximations do form the historical background of this problem, its formulation in general terms was made by Fermat in 1657. The author discusses the contributions of Lord Brouncker and of Euler toward the solution of the general case and justly credits Lagrange with the first rigorous proof of its solvability.

The relation of the Pell equation to the theory of quadratic forms is shown in connection with the more modern methods of Gauss and Dirichlet.

The excellent bibliography, together with the tabulated continued fraction developments of $\sqrt{A}$ from $A = 1{,}501$ to $A = 2{,}012$, as well as the table of fundamental solutions of the Pell equation as indicated in the sub-title, make this a very useful book for the worker in this field.

T. M. PUTNAM.

Die Berührungstransformationen: Geschichte und Invariantentheorie. Zwei Referate, der deutschen Mathematiker-Vereinigung erstattet von H. LIEBMANN und F. ENGEL. (*Jahresbericht der deutschen Mathematiker-Vereinigung.* Der Ergänzungsbände V. Band.) Leipzig, Teubner, 1914. v + 79 pp.

IN the first of these reports (pages 1–14), Liebmann gives an attractive sketch of the historical development of the theory of contact transformations. The second report (pages 15–77) on "Lie's theory of invariants of contact transformations, and its extension" by Engel is an original contribution to the theory in question rather than a report. As Engel points out in his introductory remarks, Lie took great pride in using only such methods as he had originated himself, even where simplifications due to others could be introduced to great advantage. Thus, in the second volume of his great work on transformation groups, he not only took scant notice of Mayer's simple and direct derivation of the theory of contact transformations, but ignored completely the notion of the bilinear covariant of a Pfaff differential expression, due to Lipschitz and applied with great success by Frobenius. As S. Kantor has indicated in several papers published in the *Sitzungsberichte* of the Vienna Academy (1901–03), Pfaff's problem and particularly Mayer's theory of contact transformations are greatly simplified by the use of the bilinear covariant; however, Kantor's papers, besides containing a considerable number of errors, are extremely unsystematic and obscure.

In the present work, Engel gives a very clear and simple exposition, based on the bilinear covariant, of the general theory of contact transformations and their invariants, and the application to partial differential equations, and elaborates the generalization, conceived but never worked out by Lie, of the invariant theory to Pfaff expressions in $2n$ variables. The important work of Engel should prove of great value to investigators in this field.

T. H. GRONWALL.

Veränderliche und Funktion. Von M. PASCH. Leipzig, Teubner, 1914. vi + 186 pp.

THIS little book deals primarily with the fundamental ideas at the basis of the theories of variables and functions and with their applications or illustrations by means of some of the most elementary functions. One finds a treatment of such topics as the following: order relations among numbers and the associated ideas, kinds of mathematical proof as illustrated in the foregoing discussion, point sets, sequences, variables and constants, various rational functions and classes of such functions, continuity and uniform continuity, exponential and logarithmic functions, countable sets, etc. The discussion is interspersed with a considerable number of interesting remarks belonging, one may perhaps say, to the philosophy of mathematics.

R. D. CARMICHAEL.

Ueber das Wesen der Mathematik. Zweite Auflage. Von A. Voss. Leipzig, Teubner, 1913. iv + 123 pp.

THE high conception which this author has of the character and place of mathematics is shown by the following quotation from page two: "Unsere ganze gegenwärtige Kultur . . . ihre eigentliche Grundlage in den mathematischen Wissenschaften findet." Nevertheless he believes that mathematics is the most unpopular of the sciences. The reason for this he finds in the fact that the elementary (undergraduate) courses give so small a portion of mathematical doctrine, whereas in other sciences the college student is accustomed to make far greater headway into their more vital parts.

After a brief discussion (pages 1–3) of such matters as those just mentioned the author gives a short sketch (pages 3–24) of the development of mathematics from the earliest times to the present. This is followed by a discussion (pages 24–31) of pure mathematics as a science of numbers and (pages 31–81) of the mathematical ideas of the nineteenth century. The remainder of the text (pages 81–119) is given to a variety of topics such as applications, axioms, progress, objective value of mathematics and the reforms in mathematical instruction.

R. D. CARMICHAEL.

Practical Mathematics. By NORMAN M'LACHLAN. Longmans, Green and Company, 1913. vi + 184 pp., including answers, tables and index. Price 80 cents.

"THE aim of the book is to treat those plane and solid figures with which engineers are most familiar, in such a manner that a student may make calculations on the appliances he sees and uses in daily life."

The author does not attempt to secure this result by a mathematical development of the subject, but by giving the necessary definitions and formulas and applying them in detail to a large number of examples. Methods of calculation are given very fully, so that the student who has not had experience in the manipulation of formulas and trigonometrical tables can readily use the book. In fact, the arrangement is such that the majority of students can use the book successfully without an instructor. A text of this sort fills a present need. There are many night schools and classes for boys who work part time, where the pupils desire only to be able to use mathematics as a tool and do not care for the rigorous development of formulas. For such, this book is well suited in the subject matter it presents.

We find in this book the word "trapezium" used for a quadrilateral with two sides parallel and the other two not. This follows continental rather than English usage. The encyclopedias and dictionaries do not agree in their definitions of the terms "trapezium" and "trapezoid." A uniform usage of the terms applied to the plane figures seems desirable.

The author of the book should be criticized for his selection of a title. The book has to do with practical problem solving and is not a mathematical development of the subject. Mathematics means more than substitution in formulas.

T. E. MASON.

Elementarmathematik vom höheren Standpunkte aus. Teil II: *Geometrie.* By F. KLEIN. Second edition. Leipzig, Teubner, 1913. viii + 547 pp.

IN this new edition, the text of the first edition* is reproduced with some minor changes, the literature references are brought up to date, and notes (pages 531–546) have been added on the following subjects: models illustrating affine transformations, the relation of Grassmann's geometrical conceptions to the theory of invariants, the foundations of geometry, and recent developments in the teaching of geometry.

T. H. GRONWALL.

Elementare Algebra. Akademische Vorlesungen für Studierende der ersten Semester. By EUGEN NETTO. Second edition. Leipzig, Teubner, 1913. x + 200 pp.

IN this new edition, some misprints in the first edition† have been corrected, and there are also some slight changes in the wording of various propositions. T. H. GRONWALL.

Einführung in die Vektoranalysis mit Anwendungen auf die mathematische Physik. By RICHARD GANS. Third edition. Leipzig, Teubner, 1913. x + 131 pp.

To the second edition‡ of this well-known text a chapter on tensors was added. This chapter has been entirely remodeled in the present edition, while the remaining chapters show only minor changes.

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* Reviewed by J. W. Young in BULLETIN, vol. 16 (1909–10), pp. 254–265.

† Reviewed by J. H. Tanner in BULLETIN, vol. 11 (1904–05), pp. 441–442.

‡ Reviewed by H. B. Phillips in BULLETIN, vol. 17 (1910–11), pp. 100–104.

T. H. GRONWALL.

Darstellende Geometrie des Geländes. Von RUDOLF ROTHE. Leipzig, Teubner, 1914. Small 8vo. iv + 67 pp.

THIS pamphlet, forming volume 14 of the "Mathematische Bibliothek" edited by W. Lietzmann and A. Witting, gives a very attractive presentation of that part of descriptive geometry known as the method of contour lines and familiar from its application to topographical maps. The treatment is of a quite elementary geometrical nature, and the main topics covered are: determination of distances and angles, lines of equal slope, intersections of curves and surfaces, and computation of areas and volumes. Numerous practical applications, mostly to engineering and geology, are given, and eighty-two diagrams illustrate the clear and concise text.

T. H. GRONWALL.

Le Hasard. Par EMILE BOREL. Paris, Félix Alcan, 1914. 312 pp.

THIS book belongs to the Nouvelle Collection scientifique published by Félix Alcan under the direction of Emile Borel. In it the author has given to the public the most interesting parts of his lectures and researches on the theory of probabilities. His principal aim has been to put in evidence the rôle of chance in various branches of scientific knowledge.

The work is divided into three parts. The first part contains a general exposition, remarkably free of formulas, of the principles of the theory of chance. The second part is concerned with the applications of the laws of chance to several sciences, including sociology, biology, physics and mathematics. Among the physical theories treated are some of the most recent, such as reversibility in thermodynamics and radioactivity. The third part is devoted to the philosophical basis of the laws of chance.

The book is not mathematical in its treatment. Only the most elementary mathematics, usually nothing but simple arithmetic, is needed for its reading. The discussions, however, are often illuminating. One interested in the general problems of science or in the theory of probability will find profit and pleasure in its perusal. It will furnish excellent collateral reading in connection with a course on the mathematical theory of probability.

R. D. CARMICHAEL.

An Introduction to the Mathematical Theory of Heat Conduction with Engineering and Geological Applications. By L. R. INGERSOLL and O. J. ZOBEL. Boston, Ginn and Company, 1913. vi + 171 pp.

THIS is a text on Fourier's series and heat conduction. The aim of the volume is to make the study of the subject more interesting and profitable by presenting along with the theory a large number of practical applications. These are chosen so as to make the book of direct service to geologists and engineers. Several of the practical problems which thus come in are here treated for the first time. The pulse of the concrete world throbs in every chapter and it has a healthy beat which gives one pleasure.

This book is intended for the student who has neither time nor mathematical preparation to pursue the study at great length. Consequently very little attention is given to such mathematical aspects of the theory as uniqueness, existence and convergence theorems. Hence the book will not be of special value to one interested primarily in mathematics. But its clear treatment of numerous practical applications will render it of distinct service to those for whom it was prepared, namely, students of physics and engineering who desire an elementary and brief treatment of the conduction of heat.

The arrangement of material, from a pedagogic point of view, is most excellent and deserves to be signalized with emphasis. The exposition is clear and interesting.

R. D. CARMICHAEL.

Le Système du Monde. Histoire des Doctrines cosmologiques de Platon à Copernic. Tome premier. Par PIERRE DUHEM. Paris, Hermann, 1913. 512 pp.

THE whole of the present volume is given to an account of the Hellenic cosmology beginning essentially with that of Plato. In order that the reader may understand better the doctrines of Plato and his successors a brief exposition is first given (pages 5–27) of the earlier astronomical teachings of

Pythagoras and the school founded by him. Following this is an illuminating account of the cosmology of Plato (pages 28–101) and certain developments of it among his followers (pages 102–129).

The most interesting part of the book is that (pages 130–241) in which the "physics" of Aristotle is treated.

The remaining four (of the eight) chapters deal in order with the following topics: the theories of time, space, and the void after Aristotle (pages 242–350); the dynamics of the Greeks after Aristotle (pages 351–398); the heliocentric astronomies (pages 399–426); the astronomy of excentrics and epicycles (pages 427–496).

Throughout the book the work of the author seems to have been done with care. The exposition is good. The matter brought together is of the highest interest to every student of the history of science. There is, however, but little of it which makes a special appeal to the mathematician as such; and that little consists of such beginnings of mathematical sciences as Aristotle's argument (see pages 213–214) to prove the sphericity of the surface of still water, this being the first instance (according to our author) in which mathematical reasoning has been used to establish a law of equilibrium for heavy liquids.

R. D. CARMICHAEL.

Cours de Mécanique. Vol. I. By LÉON LECORNU. Paris, Gauthier-Villars, 1914. vii+536 pp.

IN preparing the course in mechanics given at l'Ecole Polytechnique Professor Lecornu has apparently met difficulties similar to those encountered by the teachers in the technical schools of America. In the first place the time alloted to the subject is only 37 lessons for each of two years. This has made it necessary to put a portion of the kinematics of a point into the entrance requirements, and to transfer kinematics of machines to the course in geometry and the theory of the potential to the course in analysis. All of these topics are, however, discussed at length in the present volume.

In the second place the author has had to resist a demand for the teaching of practical applications with the necessary omission of much of the pure theory. He is firm in his belief that the course in l'Ecole Polytechnique should be purely theoretical. His position is stated in the preface and, among some quotations in support of it, the following from General Langlois is particularly interesting. "The officers who leave the school at the end of one year are, in general, inferior to their comrades in the matter of studying logically and deeply a scientific question of tactics or organization. The method of work indispensable to every man of action demands imperiously the study of a science to its foundations, a study which makes the intellect supple and develops a habit of logical deduction necessary to one who commands."

The first part of this volume consists of a chapter on vectors and one on displacements of a rigid body. In the second part the subject of the kinematics of a point and a solid body is presented in the usual manner. The application of kinematics to the theory of machines is quite extensive. Following a classification attributed to Willis, elementary machines are arranged in three classes according as the ratio of transmission is (*a*) constant in magnitude and sign, (*b*) variable in magnitude but constant in sign, or (*c*) variable in both magnitude and sign. Each class is subdivided into three kinds according as the transmission is by (*a*) direct contact, (*b*) a rigid intermediary, or (*c*) a flexible intermediary. The author then gives many examples and develops the theory involved in each subdivision. Statics and dynamics of a point form the subject of the third part, which includes the general principles of mechanics, the theory of the newtonian potential, the motion of a free particle in constant and central fields of force, a short section on ballistics, and the motion of a particle on a curve and on a surface. The last part is devoted to the statics of systems of bodies.

Much of the material in the first volume is taught in courses other than the one for which this text has been prepared and is included here for review or reference. Announcement has been made that the second volume is in press and the third in preparation.

The book contains no problems for solution by the student.

W. R. LONGLEY.

Problèmes de Mécanique et Cours de Cinématique. By C. GUICHARD. Paris, A. Hermann et Fils, 1913. 156 pp.

THIS little book has been edited by MM. Dautry and Deschamps and published by l'Association générale des étudiants de Paris. It represents a course given at the Sorbonne in 1912 by Professor Guichard to the candidates for the "certificat de mécanique rationnelle."

The first four chapters contain solutions of the problems proposed under the following headings: plane kinematics, kinematics of a solid body, dynamics of a point and geometry of masses, and dynamics of systems of bodies. The remainder of the book (about 70 pages) is devoted to an exposition of the theories of kinematics.

W. R. LONGLEY.

NOTES.

THE twenty-second summer meeting of the American Mathematical Society will be held at the University of California and Stanford University on Tuesday–Thursday, August 3–5. Titles and abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary by July 5.

THE March number (volume 16, number 3) of the *Annals of Mathematics* contains the following papers: "Note on normal sections of a surface in a space of n dimensions," by C. L. E. MOORE; "An algebraic treatment of the theorem of closure," by A. A. BENNETT; "An integral equation of the Volterra type," by T. H. GRONWALL; "The linear continuum in terms of point and limit," by R. L. MOORE; "A plane cubic Cremona transformation and its inverse," by F. M. MORGAN; "Relation between the roots of a rational integral function and its derivative," by FRANK IRWIN.

THE forty-third meeting of the French association for the advancement of science was held at Havre July 27–30 under the presidency of A. GAUTHIER. M. BRESSE was chairman of the mathematical section, before which the following papers were presented: "History of calculating machines," by A. GÉRARDIN; "On the periodic movement of a viscous fluid," by R. MESNY; "On Foucault's pendulum" and "The conic and sextic integrals of two homogeneous linear differential equations of the second order," by G. BRESSE; "Note in memory of Henri Poincaré," by E. LEBON; "Indeterminate analysis and factorization," by A. GÉRARDIN; "On the cycle of singular algebraic points," by M. PELLET; "New method of graphic elimination of the unknowns in a system of linear equations," by F. BOULAD; "On minimal surfaces" and "Imaginaries in the problems of infinitesimal geometry," by M. CLAPIER; "Some properties of the lemniscate of Bernoulli" and "Heronian triangles," by E. N. BARISIEN; "Tables of quotients to the base 300 constructed from progressions according to the last figure of the numbers," by C. BOULOGNE; "Integration and measure in abstract assemblages," by M. FRÉCHET; "The theorem of Dirichlet on the arithmetic progression," by L. AUBRY; "Elementary remarks on descriptive geometry and perspective," by C. HALPHEN; "Geometry in the teaching of mathematics," by J. RICHARD; "Developables of algebraic curves" and "Heronian triangles," by M. WELSCH.

A MEETING of the London mathematical society was held January 14. The following papers were read: By H. H. MACDONALD, "A class of diffraction problems"; by H. E. J. CURZON, "Halphen's transformation"; by A. YOUNG, "A Christmas problem in probabilities"; by E. H. BERWICK, "The condition that a quintic equation should be solvable by radicals"; by J. LARMOR, "The variation of the earth's angular velocity of rotation."

THE following parts of the Encyklopädie der mathematischen Wissenschaft are in the press, and may be expected in a few weeks: II C 2, Graphical and numerical quadrature and graphical and numerical integration of ordinary and partial differential equations, by C. RUNGE and F. A. WILLERS; II C 3, Recent investigations of functions of real variables, by E. BOREL, M. FRÉCHET, P. MONTEL, and L. ZORETTI. III 1, Elementary geometry and elementary non-euclidean geometry (conclusion), by M. ZACHARIAS; III 1, Geometry of the triangle, by G. BERKHAN (deceased); III 2, Special plane algebraic curves, by G. LORIA; III 3, Fundamental properties of algebraic surfaces, by G. CASTELNUOVO and F. ENRIQUES; III 3, Algebraic surfaces from the standpoint of birational transformations, by G. CASTELNUOVO and F. ENRIQUES; III 3, Contact transformations, by H. LIEBMANN; III 3, Geometric theory of differential equations, by H. LIEBMANN; V 3, Optics of waves, by M. v. LAUE; VI 1 B, Terrestrial magnetism, by A. SCHMIDT; VI 2, Theory of the planets, by K. F. SUNDMAN; VI 2, Theory of the earth's moon, by E. W. BROWN.

The general editorship of volume II, analysis, will be conducted by Professor R. FRICKE.

Work on the publication of the French edition has been suspended.

THE mathematical society of Amsterdam has just published the first of the two volumes of the complete edition of the works of T. J. STIELTJES, containing the papers of the years 1876–1886.

THE second part of each volume of the second German edition of Pascal's Repertorium der höheren Mathematik has long been announced as in press. It is now expected to publish both parts during the next few months.

WILEY and Sons, of New York, announce that HENDERSON'S "Statistics and laws of mortality" will be published in June of this year.

THE United States Bureau of Education has just issued a new bulletin of the International commission on the teaching of mathematics. This is entitled Curricula in Mathematics and has been prepared by MR. J. C. BROWN, of the Horace Mann School, New York City. It sets forth the work done in each school year of the standard type of course in each of the leading countries of the world. Copies may be obtained by addressing the Bureau of Education at Washington.

MILAN TECHNICAL SCHOOL. The following courses in mathematics are being given during the present year (November 24, 1914–June 30, 1915):—By Professor A. JORINI: Analytic geometry, four hours.—By Professor U. CISOTTI: Mathematical analysis, seven hours (for engineers).—By Professor G. TOMASELLI: Mathematical analysis, four hours (for architects).—By Professor C. CAPELLI: Projective and descriptive geometry, nine hours.—By Professor M. ABRAHAM: Rational mechanics, five hours.—By Professor G. FORNI: Rational mechanics, three hours (second term, for architects). All these subjects must have been completed before a student can register in the major three year course.

Professor G. JUNG has been made professor emeritus.

PROFESSORS A. SCHOENFLIES and L. BIEBERBACH have been appointed professors of mathematics in the University of Frankfurt.

PROFESSOR E. SALKOWSKI, of the technical school at Berlin, has been appointed professor of mathematics in the technical school of Hanover.

DR. O. DAUZER has been appointed docent in projective and descriptive geometry at the technical school of Vienna.

MR. L. G. OWEN has been appointed professor of mathematics in the Government College at Rangoon.

PROFESSOR A. D. PITCHER, of Dartmouth College, has been appointed professor and head of the department of mathematics in Adelbert College of Western University.

DRS. EDWARD KIRCHER and G. A. PFEIFFER have been appointed to the recently established Benjamin Peirce instructorships at Harvard University for the year 1915–16.

DR. W. DEIMLER, of the technical school at Munich, was killed in battle, at the age of 30 years.

PROFESSOR G. HOLZMÜLLER, of the industrial school at Hagen, died November 27, at the age of 70 years.

DR. A. LACKNER, of the technical school at Vienna, was killed in battle in November, at the age of 29 years.

PROFESSOR G. PIRONDINI, of the technical institute of Rome died January 17, at the age of 57 years.

PROFESSOR S. W. SHATTUCK, of the University of Illinois, died on February 13, at the age of 74 years. Professor Shattuck was connected with the mathematical department as professor and head of the department from 1868 to his retirement from active service in 1912. He had been a member of the American Mathematical Society since 1891.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BAERWALD (H.). Ueber das Wesen der Naturgesetze. Antrittsvorlesung. Heidelberg, G. Koester, 1914.

BERANEK (A.). Zur sphärischen Abbildung der Flachèn zweiter Ordnung und ihrer stereographischen Projektion. Baden b. Wien, 1913. Gr. 8vo. 12 pp.

BETSCHLER (F.). Ueber Integraldarstellungen, welche aus speziellen Randwertproblemen bei gewöhnlichen linearen inhomogenen Differentialgleichungen entspringen. (Diss.) Wurtzburg, 1914.

BIANCHI (L.). Lezioni di geometria analitica. Pisa, E. Spoerri (F. Mariotti), 1915. 8vo. 4+604 pp. L. 22.00

BOLDT (A.). Das Ende des Fermatschen Spukes. Allgemeiner und wahrhaft wunderbarer Beweis zu der Behauptung: Unmöglich sind Gleichungen nach $x^\lambda + y^\lambda = z^\lambda$ (x, y, z, ganze Zahlen, $\lambda > 2$). Grosser Fermatscher Satz. Naumburg, 1914. 8vo. 16 pp. M. 1.80

BOULIGAUD (G.). Sur les fonctions de Green et de Neumann du cylindre. Paris, 1914. 4to. 81 pp.

CHAPELON (J.). Sur les relations entre les nombres des classes de formes quadratiques binaires de déterminant négatif. Paris, 1914. 4to. 203 pp.

COLEMAN (P.). Coordinate geometry: an elementary course. Oxford, Clarendon Press, 1915. 8vo. 4s. 6d.

CONTENSON (L. DE). La certitude mathématique. Les fondements mathématiques dans l'hypothèse de la philosophie critique (système cartéso-kantien). Paris, Gauthier-Villars, 1914. 8vo. 93 pp. Fr. 3.25

CORRIDORE (F.). Elementi di calcolo infinitesimale, per gli studiosi di statistica. Roma, E. Loescher e C.: W. Regenberg e C. (s. tip.), 1914. 8vo. L. 2.00

CRUWELL (E. R.). Der Satz des Fermats. 3te durch den arithmetischen Beweis des Satzes vermehrte Auflage. Berlin, 1914. Gr. 8vo. 14 pp. M. 2.40

DINGLER (H.). Das Prinzip der logischen Unabhängigkeit in der Mathematik zugleich als Einfuhrung in die Axiomatik. Munchen, T. Ackermann, 1915. Gr. 8vo. 8+164 pp. Geh. M. 5.00

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L R RY

THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and seventy-fifth regular meeting of the Society was held in New York City on Saturday, February 27, 1915. The attendance at the two sessions included the following thirty-nine members:

Professor Joseph Bowden, Professor E. W. Brown, Dr. T. H. Brown, Professor B. H. Camp, Dr. Emily Coddington, Professor F. N. Cole, Dr. G. M. Conwell, Professor Elizabeth B. Cowley, Dr. Louise D. Cummings, Dr. H. B. Curtis, Professor L. P. Eisenhart, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor W. H. Garrett, Professor O. E. Glenn, Professor C. C. Grove, Professor G. H. Hallett, Professor H. E. Hawkes, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. J. K. Lamond, Mr. P. H. Linehan, Professor James Maclay, Dr. H. F. MacNeish, Dr. E. J. Miles, Mr. G. W. Mullins, Dr. G. A. Pfeiffer, Dr. H. W. Reddick, Professor R. G. D. Richardson, Mr. P. R. Rider, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor Edwin R. Smith, Professor Oswald Veblen, Mr. R. A. Wetzel, Miss E. C. Williams.

The President of the Society, Professor Ernest W. Brown, occupied the chair, being relieved at the afternoon session by Vice-President Oswald Veblen. The Council announced the election of the following persons to membership in the Society: Professor J. V. Balch, Bethany College; Professor E. J. Berg, Union College; Mr. Millar Brainard, Chicago, Ill.; Mr. L. C. Cox, Purdue University; Mr. C. H. Forsyth, University of Michigan; Dr. H. C. Gossard, University of Oklahoma; Mr. M. S. Knebelman, Lehigh University; Dr. W. V. Lovitt, Purdue University; Dr. L. C. Mathewson, Dartmouth College; Mr. A. L. Miller, University of Michigan; Dr. Bessie I. Miller, Johns Hopkins University; Mr. I. R. Pounder, University of Toronto; Mr. L. L. Steimley, Indiana University; Mr. Chid-Cheow Yen, Tangshan Engineering College. Three applications for membership in the Society were received.

From the early days of the Society the informal dinners held in connection with the meetings have been a most valuable supplement to the scientific programme, furnishing opportunities for a general exchange of views, conferences on questions of mathematics and of general policies, and renewing old and making new acquaintance. To promote this social side of the Society's activities, the Council has recently appointed a committee, consisting of Professors Fiske, Hawkes, and Kasner, to arrange a dinner for each New York meeting and by giving publicity to these occasions to induce a large attendance. Notice of the dinner is sent out with the programme of the meeting, and members expecting to attend are requested to fill out the accompanying card and return it to the Secretary, in order that adequate accommodations may be provided. The result at the recent annual meeting was the large attendance of seventy members. The February meeting is always, for several reasons, a smaller affair. Yet on this occasion nineteen members spent a very enjoyable evening together.

The following papers were read at this meeting:

(1) Professor M. FRÉCHET: "Sur les fonctionnelles bilinéaires."

(2) Professor A. S. HATHAWAY: "Gamma coefficients."

(3) Mr. P. H. LINEHAN: "Equilong invariants of irregular and regular analytic curves."

(4) Professor B. H. CAMP: "Multiple integrals over infinite fields."

(5) Mr. A. R. SCHWEITZER: "On the methods of mathematical discovery."

(6) Mr. P. R. RIDER: "An extension of Bliss's form of the problem of the calculus of variations, with applications to the generalization of angle."

(7) Professor E. B. WILSON: "The Ziwet-Field note on plane kinematics."

(8) Professor O. E. GLENN: "Ternary transvectant systems."

(9) Dr. E. J. MILES: "Note on the application of the calculus of variations to a problem in mechanics."

(10) Professor A. B. FRIZELL: "The permutations of the natural numbers cannot be well ordered."

(11) Mr. C. H. FORSYTH: "Osculatory interpolation formulas."

(12) Mr. J. F. RITT: "A function of a real variable with any desired derivatives at a point."

(13) Mr. J. F. RITT: "On Babbage's functional equation."

Professor Fréchet's paper was communicated to the Society through Professor D. R. Curtiss. Mr. Ritt was introduced by Professor Kasner. In the absence of the authors the papers of Professor Fréchet, Professor Hathaway, Mr. Schweitzer, Professor Wilson, Professor Frizell, and Mr. Forsyth were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Fréchet investigates operations $U_{f,\phi}$ which are linear with respect to f and to ϕ separately. Generalizing a known theorem of F. Riesz, he proves that $U_{f,\phi}$ may be represented as follows:

$$U_{f,\phi} = \int_a^{a'} \int_b^{b'} f(s)\phi(t)d_s d_t u(s, t),$$

the double integral on the right being defined as the limit of

$$\Sigma_i \Sigma_j f(\xi_i)\phi(\eta_j)\Delta_{i,j}u,$$

where

$$\Delta_{i,j}u = u(s_i, t_j) - u(s_i, t_{j-1}) - u(s_{i-1}, t_j) + u(s_{i-1}, t_{j-1}).$$

A striking result is that the function $u(s, t)$ need fulfill no other condition than that $|\Sigma_i \Sigma_j \epsilon_i \epsilon_j' \Delta_{i,j}u|$ have a finite upper bound whatever may be the signs of ϵ_i, ϵ_j', provided $|\epsilon_i| = |\epsilon_j'| = 1$. This representation by generalized double integrals is then applied to second differentials of continuous operations.

2. Professor Hathaway defines a gamma coefficient of coordinates $x, y, \cdots$ and parameters $a, b, \cdots$. The coefficient of two dimensions is

$$[axby] = (ax + by)\Gamma(x + y)/\Gamma(x + 1)\Gamma(y + 1),$$

and similarly for any dimension of coordinates.

This coefficient is shown to be constant upon an axis and, at any point, equal to the sum of its values at the points which precede it by a unit in the direction of each axis.

Taking integral values, none negative, for coordinates and parameters, an integral function is formed of corresponding arguments $p, q, \cdots$, consisting of the sum of all terms with coefficients of a given weight $n = ax + by + \cdots$, the coordinates of a coefficient being the exponents of the corresponding arguments. If m be the greatest parameter, it is shown that this function equals the sum of the nth powers of the roots of the equation

$$x^m = px^{m-a} + qx^{m-b} + \cdots.$$

This is a generalization of the formula for s_n given by Waring and proved by Serret (Cours d'Algèbre supérieure, volume I, page 445, article 196) from the formula of Lagrange. Waring's formula is deduced from the above by taking a, $b, \cdots, m$, as the natural numbers $1, 2, \cdots, m$. When some parameters are equal, the general formula obtained above cannot be deduced from that of Waring. For example, the above result becomes the multinomial theorem for $a = b = \cdots = m = 1$; but Waring's formula, in the same case, is $(p + q + \cdots)^n = (p + q + \cdots)^n$.

Further properties of these coefficients will be the subject of a future paper.

3. Irregular analytic curves

$$v = \alpha_q u^{q/p} + \alpha_{q+1} u^{(q+1)/p} + \alpha_{q+2} u^{(q+2)/p} + \cdots$$

(u, v being Hessian line coordinates) are shown by Mr. Linehan to possess invariants under the group of equilong transformations of the plane except when $p = 2$. For each of the three cases which arise when invariants exist, the simplest invariant is derived.

The simplest invariant of a regular analytic curve under the linear equilong transformations of the plane is also obtained.

4. Professor Camp's immediate object in this paper is to prepare a foundation for a discussion, to be given later, of multiple and iterated integrals containing parameters, in which the integrations are extended over infinite fields. For this purpose it is necessary to coordinate the various definitions of these multiple integrals which have been given in recent years, to consider more fully than has been done heretofore the conditions of their existence, their relation to the iterated integrals, and certain of their fundamental properties.

5. In a previous article* Mr. Schweitzer stated a heuristic

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* *Revue de Métaphysique et de Morale*, March, 1914.

principle of comparison based on identity, or principle of unification of terms. In the present paper is stated a principle of comparison antithetic to the preceding and based on diversity. The latter is called the principle of "furcation" of terms and is stated as follows:

The existence of dissimilarities between given terms implies the existence of dissimilar general terms which underlie or embrace the particular terms.

It is important to recognize the interdependence of the two principles. Thus by replacing "dissimilarities," "dissimilar" by "similarities" (resemblances), "similar" respectively, one obtains a possible interpretation of the author's principle of unification. The principle of "furcation" is useful in providing, directly or indirectly, problems of conflict for solution by means of the principle of unification.

An interesting special case of the principle of furcation is provided by equivalent mathematical theories or concepts which diverge under generalization. An instance of this kind is the author's "trifurcative" generalization* of the betweenness relation in the foundations of geometry. Many other illustrations might be given.

A concluding part of the author's paper is devoted to a critique of a remarkable article by J. T. Merz, History of European Thought of the Nineteenth Century, volume II, pages 627–740: "Development of mathematical thought." As a critical basis, the author's article on the working hypotheses of mathematics (cited above) is used.

6. Bliss has developed a theory of the calculus of variations for integrals of the form $I = \int f(x, y, \tau)ds$, where $\tan \tau$ is the slope of the curve considered. Mr. Rider's paper extends the theory to the integral $I = \int f(x, y, z, \tau, \sigma)ds$, where σ is the angle that a space curve makes with its own projection in the xy-plane, and τ is the angle that this projection makes with the x-axis. Necessary and sufficient conditions are studied.

A transversal surface is used in giving certain generalized definitions of angle and of solid angle. Incidentally, the differential equations of geodesics and generalized geodesics on this surface are derived. The definitions of angle reduce under proper conditions to the definitions used in euclidean geometry.

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* *American Journal*, vol. 31 (1909), p. 366.

7. Referring to the adoption by Ziwet and Field of the operators i and $e^{\phi i}$ of Burali-Forti and Marcolongo, Professor Wilson points out (1) that these operators are not algebraic as their form indicates, (2) that the operator $e^{\phi \mathrm{u} \wedge}$, which is patently non-algebraic, will not only serve the same purposes, but in addition lead to the derivation by A. C. Lunn of the Euler-Rodrigues form for rotations, and (3) that the algebraic operator $e^{\phi'}$, used in connection with the Gibbs bivector, will also accomplish the ends desired by Ziwet and Field. This note will appear in the *American Mathematical Monthly.*

8. The methods for the construction of fundamental systems of invariant formations of ternary quantics, known up to the present time, are tentative. The purpose of Professor Glenn's paper is, first, to deduce the simplest possible algorithm for the construction of such systems. Instead of deriving all invariants by means of one transvectant operation, four such operations are introduced, which may, however, be united under one major process of making cogredient substitutions in a double mixed polar. The main theorem of the paper is: If two systems of forms are finite and complete, the system derived therefrom by the above major transvectant process is finite and complete., A method of deriving the irreducible members of such a system is given. By the theorems of the paper the twenty invariant formations of two conics, for instance, may be readily written down.

9. The problem considered by Dr. Miles is: Given a chord of definite length and variable density; to find its form for uniform horizontal distribution of the mass if the center of gravity lies as low as posssible.

The density is assumed to be a function of x, say $r(x)$, and a problem in variations of the Lagrangian type results. The functions y and r are then determined and the curve is found to be a parabola—the solution given in text books on mechanics.

10. By methods developed in previous papers read before the Society (Lincoln, November, 1914; Chicago, December,

1914, and New York, January, 1915), Professor Frizell shows that the assumption that the set of permutations of the natural numbers can be well ordered leads, by a line of reasoning due to Cantor, to two sets of permutations which are both in one-to-one correspondence with the whole set of permutations of the natural numbers but cannot be put into one-to-one correspondence with each other.

11. As is well known, whenever successive intervals are to be interpolated, separate curves are used in the several intervals. In osculatory interpolation these separate curves are required to have the same slopes and curvatures at their points of intersection for purposes of smoother graduation. The appropriate formulas to be used in interpolating several values in each interval depend upon the fundamental interpolation curve upon which they are based, and formulas have been derived based upon Newton's and Everett's interpolation formulas. Mr. Forsyth derives the formulas based upon Stirling's and Bessel's formulas together with a series of corrections of fifth differences to be used in connection with the four formulas in lieu of the formulas in succeeding intervals. The paper will appear in the *Quarterly Publications* of the American Statistical Association.

12. There exists no analytic function having the sequence of numbers $a_1, a_2, \cdots, a_n,$ as derivatives for $x = 0$, unless $| a_n/n! |^{1/n}$ stays bounded as n increases indefinitely.

Professor Kasner suggested the desirability of learning whether a non-analytic function can be found when no analytic function exists. Mr. Ritt has shown that the function

$$\sum_{n=1}^{n=\infty} \frac{a_n x^n}{n!}\left(1 - e^{-1/(b_n x^2)}\right),$$

where $1 < b_n > | a_n |$, has the desired sequence of derivatives.

13. About 1815, Charles Babbage attempted to find all functions of which the nth iterative is the independent variable itself. He observed that if $f(x)$ is such a function, $\varphi^{-1}f\varphi(x)$ will also be one, where $\varphi(x)$ is arbitrary; but he had no means of knowing how general his solution was.

Mr. Ritt has commenced a rigorous discussion of Babbage's functions, limiting himself to the case of the real variable, and making a set of five assumptions. It appears that the most general solution, when n is greater than 2, is $\varphi^{-1}f^{r}\varphi(x)$, where the integer r is prime to n. The case $n = 2$ is discussed separately and a simple algorism is given for reducing all differentiable functions of order 2 to a single type.

F. N. Cole,
Secretary.

THE LEGENDRE CONDITION FOR A MINIMUM OF A DOUBLE INTEGRAL WITH AN ISOPERIMETRIC CONDITION.

BY DR. CHARLES ALBERT FISCHER.

(Read before the American Mathematical Society, February 28, 1914.)

The Legendre, or second necessary, condition for a minimum of a double integral, where there is no isoperimetric condition, has been derived by Kobb,* where the equations of the surfaces involved are in parametric form, and by Mason,† where x and y are the independent variables. The analogous condition for the isoperimetric problem has been proved to be sufficient to insure a permanent sign to the second variation,‡ but it has not been proved to be necessary.

In the present paper this condition,

$$h_{pp}(x, y, z, p, q; \lambda)h_{qq}(x, y, z, p, q; \lambda) - h_{pq}^2(x, y, z, p, q; \lambda) \geqq 0,$$

or expressed in parametric form,

$$H_{11}(x, y, z, x_u, x_v, \cdots, z_v; \lambda)H_{22}(x, y, z, x_u, x_v, \cdots, z_v; \lambda) - H_{12}^2(x, y, z, x_u, x_v, \cdots, z_v; \lambda) \geqq 0,$$

is proved to be necessary for either a maximum or a minimum.

Given two functions $f(x, y, z, p, q)$ and $g(x, y, z, p, q)$ and a surface

S: $$z = z(x, y)$$

---

* "Sur les maxima et les minima des intégrales doubles," *Acta Mathematica*, vol. 16 (1892), p. 108.

† "A necessary condition for an extremum of a double integral," Bulletin, vol. 13 (1907), p. 293.

‡ Kobb, *Acta Mathematica*, vol. 17 (1893), p. 331.

which satisfies the Lagrange differential equation

$$(1)\qquad h_z(x, y, z, p, q; \lambda) - \frac{\partial}{\partial x} h_p(x, y, z, p, q; \lambda) - \frac{\partial}{\partial y} h_q(x, y, z, p, q; \lambda) = 0;^*$$

it is desired to find a second condition which must be satisfied if the surface S gives a value to the integral

$$(2)\qquad J = \int\int_\Omega f(x, y, z, p, q)dxdy$$

as small as that given by any other admissible surface in the neighborhood of S. The region Ω is assumed to be bounded by a curve L of class D',† without double points, and the number of its intersections with any line parallel to either of the axes is assumed to be less than a fixed constant. The function $z(x, y)$ is assumed to be of class C'' in Ω, as are also the functions f and g in the neighborhood of S. A surface is said to be admissible if it is of class D', intersects S along a space curve which projects into L, and gives the same value as S to the double integral

$$K = \int\int_\Omega g(x, y, z, p, q)dxdy.$$

It will also be assumed that $z(x, y)$ is not a solution of the equation

$$(3)\qquad g_z(x, y, z, p, q) - \frac{\partial}{\partial x} g_p(x, y, z, p, q) - \frac{\partial}{\partial y} g_q(x, y, z, p, q) = 0.$$

If a one parameter family of admissible surfaces

$$\bar{S}: \qquad z = z(x, y) + \epsilon\bar{\zeta}(x, y, \epsilon)$$

is given, where the function $\bar{\zeta}(x, y, 0)$ and its partial derivative $\zeta_\epsilon(x, y, 0)$ are of class D', and this value substituted for z in the function f, the first variation of J must vanish because of equation (1). The second variation is found to be

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* Bolza: Vorlesungen über Variationsrechnung, p. 662.

† Bolza, loc. cit., p. 63.

$$
\begin{aligned}
(4)\qquad \delta^2 J = \epsilon^2 \int\!\!\int_\Omega (f_{zz}\bar{\zeta}^2 + 2f_{zp}\bar{\zeta}\bar{\zeta}_x + 2f_{zq}\bar{\zeta}\bar{\zeta}_y + f_{pp}\bar{\zeta}_x^2 \\
+ 2f_{pq}\bar{\zeta}_x\bar{\zeta}_y + f_{qq}\bar{\zeta}_y^2 + f_z\bar{\zeta}_\epsilon + f_p\bar{\zeta}_{x\epsilon} + f_q\bar{\zeta}_{y\epsilon})dxdy.
\end{aligned}
$$

Since the surfaces considered are admissible, $\delta^2 K$, which can be evaluated in the same way, vanishes. If it is multipled by λ and added to $\delta^2 J$, and then Green's theorem* is applied, equation (4) becomes

$$
\begin{aligned}
\delta^2 J = \epsilon^2 \int\!\!\int_\Omega (h_{zz}\bar{\zeta}^2 + 2h_{zp}\bar{\zeta}\bar{\zeta}_x + \cdots + h_{qq}\bar{\zeta}_y^2)dxdy \\
+ \epsilon^2 \int\!\!\int_\Omega \left(h_z - \frac{\partial}{\partial x} h_p - \frac{\partial}{\partial y} h_q\right)\bar{\zeta}_\epsilon\, dxdy,
\end{aligned}
$$

where as usual $h = f + \lambda g$. The last integral vanishes on account of equation (1), leaving

$$
(5)\qquad \delta^2 J = \epsilon^2 \int\!\!\int_\Omega (h_{zz}\bar{\zeta}^2 + 2h_{zp}\bar{\zeta}\bar{\zeta}_x + \cdots + h_{qq}\bar{\zeta}_y^2)dxdy.
$$

It will now be proved that if there is a point on S where the inequality

$$
(6)\qquad h_{pp}h_{qq} - h_{pq}^2 < 0
$$

is satisfied, the function $\bar{\zeta}(x, y, \epsilon)$ can be chosen in such a way that $\delta^2 J$ will be negative, and consequently there is no minimum.

If there is such a point there must be a region including the point where inequality (6) is satisfied. Two distinct points P_0 and P_0' in such a region can then be chosen, whose coordinates will be called $x_0, y_0, z(x_0, y_0)$ and $x_0', y_0', z(x_0', y_0')$. Since $z(x, y)$ is not a solution of equation (3), it can be assumed that

$$
\begin{aligned}
(7)\qquad g_z(x_0', y_0', \cdots) - \frac{\partial}{\partial x_0'} g_p(x_0', y_0', \cdots) \\
- \frac{\partial}{\partial y_0'} g_q(x_0', y_0', \cdots) \neq 0.
\end{aligned}
$$

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* Bolza, loc. cit., p. 654.

The expression

$$h_{pp}(x_0, y_0, \cdots) \cos^2 \alpha + 2h_{pq}(x_0, y_0, \cdots) \cos \alpha \sin \alpha + h_{qq}(x_0, y_0, \cdots) \sin^2 \alpha$$

must vanish for two positive values of α less than π, and changes its sign when and only when α passes through one of them. It follows that a positive constant k^2 and a finite interval can be chosen in such a way that if α_1 and α_2 are any two angles in the interval, the inequalities

$$(8) \quad h_{pp} \cos^2 \alpha_i + 2h_{pq} \cos \alpha_i \sin \alpha_i + h_{qq} \sin^2 \alpha_i < -k^2 \qquad (i = 1, 2),$$

are satisfied at P_0, and since h_{pp}, h_{pq} and h_{qq} are continuous, they are satisfied at every point of S in a neighborhood of P_0. If there is given any positive constant δ it is possible to select two distinct angles α_1' and α_2' near to a root of the equation

$$h_{pp}(x_0', y_0', \cdots) \cos^2 \alpha + 2h_{pq}(x_0', y_0', \cdots) \cos \alpha \sin \alpha + h_{qq}(x_0', y_0', \cdots) \sin^2 \alpha = 0,$$

such that the inequalities

$$(9) \quad | h_{pp} \cos^2 \alpha_i' + 2h_{pq} \cos \alpha_i' \sin \alpha_i' + h_{qq} \sin^2 \alpha_i' | < \delta \qquad (i = 1, 2),$$

are satisfied at every point of S in a neighborhood of P_0'. For convenience these angles will be chosen so that $\alpha_2 - \alpha_1 = \alpha_2' - \alpha_1'$.

Two rhombuses R and R' will be defined as follows:* R is bounded by the lines

$$d - u_1 = 0, \quad d - u_2 = 0, \quad d + u_1 = 0, \quad d + u_2 = 0,$$

where

$$u_i = (x - x_0) \cos \alpha_i + (y - y_0) \sin \alpha_i, \quad (i = 1, 2),$$

and R' by the lines

$$d - u_1' = 0, \quad d - u_2' = 0, \quad d + u_1' = 0, \quad d + u_2' = 0,$$

where

$$u_i' = (x - x_0') \cos \alpha_i' + (y - y_0') \sin \alpha_i'.$$

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* Compare with Mason, loc. cit., p. 295.

The constant d will be taken so small that R and R' do not overlap, and are entirely within the projections on the x, y-plane of the respective neighborhoods in which the inequalities (8) and (9) are satisfied. A function $\zeta(x, y)$ will be defined as identically zero outside of R, and inside of R as equal to the distance of the point x, y from the nearest side of R. That is,

$$\zeta(x, y) = d \mp u_i, \qquad (i = 1, 2),$$

where the sign and subscript of u are chosen so as to make ζ as small as possible. The function $\zeta'(x, y)$ will be chosen in the analogous way. Since $\alpha_2 - \alpha_1 = \alpha_2' - \alpha_1'$ the rhombuses R and R' are of the same size and shape and the equation

$$\int\int_R \zeta(x, y)\,dxdy = \int\int_{R'} \zeta'(x, y)\,dxdy$$

is satisfied.

The function $\bar{\zeta}(x, y, \epsilon)$ will now be defined by the equation

$$\epsilon\bar{\zeta}(x, y, \epsilon) = \epsilon\zeta(x, y) + \epsilon'(\epsilon)\zeta'(x, y),$$

where $\epsilon'(\epsilon)$ is to be determined by the condition that the surfaces $\bar{S}$ be admissible. The first variation of K is found to be equal to

$$\int\int_R \left(g_z - \frac{\partial}{\partial x} g_p - \frac{\partial}{\partial y} g_q\right) \zeta\,dxdy$$
$$+ \frac{d\epsilon'(0)}{d\epsilon} \int\int_{R'} \left(g_z - \frac{\partial}{\partial x} g_p - \frac{\partial}{\partial y} g_q\right) \zeta'\,dxdy = 0.$$

If the mean value theorem is applied to these integrals and the equation is solved for $d\epsilon'(0)/d\epsilon$, it becomes

$$\frac{d\epsilon'(0)}{d\epsilon} = - \frac{\left(g_z - \dfrac{\partial}{\partial x} g_p - \dfrac{\partial}{\partial y} g_q\right)_{x=\xi,\, y=\eta}}{\left(g_z - \dfrac{\partial}{\partial x} g_p - \dfrac{\partial}{\partial y} g_q\right)_{x=\xi',\, y=\eta'}},$$

where ξ, η is a point in R and ξ', η' a point in R'. The denominator cannot vanish if d is sufficiently small, because of inequality (7). Consequently a finite constant m can be chosen such that

$$\left|\frac{d\epsilon'(0)}{d\epsilon}\right| < m.$$

If the second variation of K is equated to zero, the coefficient of $d^2\epsilon'(0)/d\epsilon$ is also equal to

$$\int\int_{R'}\left(g_z-\frac{\partial}{\partial x}g_p-\frac{\partial}{\partial y}g_q\right)\zeta' dxdy \neq 0.$$

Consequently this second derivative exists. Therefore the functions

$$\bar{\zeta}(x,\, y,\, 0)=\zeta(x,\, y)+\frac{d\epsilon'(0)}{d\epsilon}\,\zeta'(x,\, y),$$
$$\zeta_\epsilon(x,\, y,\, 0)=\frac{d^2\epsilon'(0)}{d\epsilon^2}\,\zeta'(x,\, y)$$

exist and they have the required continuity.

When $\bar{\zeta}(x,\, y,\, \epsilon)$ is determined in this way equation (5) may be written

$$\delta^2 J=\epsilon^2\int\int_{R+R'}(h_{zz}\bar{\zeta}^2+2h_{zp}\bar{\zeta}\bar{\zeta}_x+2h_{zq}\bar{\zeta}\bar{\zeta}_y)dxdy$$
$$+\epsilon^2\int\int_R(h_{pp}\cos^2\alpha_i+2h_{pq}\cos\alpha_i\sin\alpha_i+h_{qq}\sin^2\alpha_i)dxdy$$
$$+\epsilon^2\left(\frac{d\epsilon'(0)}{d\epsilon}\right)^2\int\int_{R'}(h_{pp}\cos^2\alpha_i'+2h_{pq}\cos\alpha_i'\sin\alpha_i'$$
$$+h_{qq}\sin^2\alpha_i')dxdy,$$

where i takes the values 1 and 2 in the appropriate parts of R and R'. It can now be easily proved that

$$\delta^2 J<-\epsilon^2 A(k^2-Md(d+4)(1+m^2)-\delta m^2),$$

where M is the largest of the maxima of the numerical values of h_{zz}, h_{zp} and h_{zq}, and A is the area of R.* Since d and δ can be taken as small as is desired without affecting k^2, they can be taken so small that $\delta^2 J$ will be negative and there is no minimum.

In a similar way it can be proved that there is no maximum if inequality (6) is satisfied, and the following theorem will be proved:

A necessary condition that the surface S furnish either a maximum or a minimum for the double integral (2), *relative to*

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* Compare with Mason, loc. cit., pp. 295–6.

the admissible surfaces, is that

$$(10)\quad \begin{aligned} h_{pp}(x, y, z, p, q;\ \lambda)h_{qq}(x, y, z, p, q;\ \lambda) \\ - h_{pq}^2(x, y, z, p, q;\ \lambda) \geqq 0, \end{aligned}$$

at every point of S.

If the surface S is represented by the parametric equations

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

the functions f and g must be replaced by $F(x, y\ z, x_u, x_v, \cdots z_v)$ and $G(x, y, z, x_u, x_v, \cdots, z_v)$ respectively. Then if the equations of $\overline{S}$ are written

$$x = x(u, v) + \epsilon\xi(u, v, \epsilon), \quad y = y(u, v) + \epsilon\eta(u, v, \epsilon),$$
$$z = z(u, v) + \epsilon\zeta(u, v, \epsilon),$$

the second variation becomes

$$(11)\quad \delta^2 J = \epsilon^2 \int\int_\Omega (H_{xx}\xi^2 + 2H_{xy}\xi\eta + \cdots + H_{z_v z_v}\zeta_v^2)dudv.$$

The values of J and K are assumed to be unchanged by any change in the parametric representation of S which leaves the surface S itself invariant. This furnishes a number of relations between the partial derivatives of F and G, among which are the following:*

$$(12)\quad \begin{aligned} F_{x_u x_u} = F_{11}X^2, \quad F_{x_u y_u} = F_{12}XY, \quad F_{x_u x_v} = F_{12}X^2, \\ F_{x_u y_v} + F_{x_v y_u} = 2F_{12}XY, \quad F_{x_v x_v} = F_{22}X^2, \quad F_{x_v y_v} = F_{22}XY, \end{aligned}$$

and the other formulas derived from these by permuting the letters x, y, z and X, Y, Z in the same way. The functions F_{11}, F_{12} and F_{22} are continuous and X, Y and Z are the direction cosines of the normal to S.

It will now be assumed that there is a function $\omega(u, v, \epsilon)$, such that

$$\xi(u, v, \epsilon) = \omega(u, v, \epsilon)X,$$
$$\eta(u, v, \epsilon) = \omega(u, v, \epsilon)Y,$$
$$\zeta(u, v, \epsilon) = \omega(u, v, \epsilon)Z.$$

If these values are substituted in the integrand of equation (11), it becomes a quadratic form in ω, ω_u, ω_v. The coefficient of ω_u^2 is seen to be

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* Kneser: Lehrbuch der Variationsrechnung, p. 282.

$$H_{x_ux_u}X^2 + 2H_{x_uy_u}XY + H_{y_uy_u}Y^2 + 2H_{x_uz_u}XZ \\ + 2H_{y_uz_u}YZ + H_{z_uz_u}Z^2.$$

Equations (12), with F replaced by H, reduce this to the form

$$H_{11}(X^2 + Y^2 + Z^2)^2 = H_{11}.$$

Similarly the coefficients of $\omega_u\omega_v$ and ω_v^2 can be proved equal to $2H_{12}$ and H_{22} respectively. The other coefficients will be called H_{00}, $2H_{01}$ and $2H_{02}$ respectively, and equation (11) becomes

$$\delta^2 J = \epsilon^2 \int\int_\Omega (H_{00}\omega^2 + 2H_{01}\omega\omega_u + 2H_{02}\omega\omega_v + H_{11}\omega_u^2 \\ + 2H_{12}\omega_u\omega_v + H_{22}\omega_v^2)dudv.$$

This equation is in the same form as equation (5), and from this point on the argument is so nearly the same as in the non-parametric case that it need not be repeated here. The analogue of inequality (10) is seen to be

$$H_{11}(x, y, z, x_u, \cdots, z_v; \lambda)H_{22}(x, y, z, x_u, \cdots, z_v; \lambda) \\ - H_{12}^2(x, y, z, x_u, \cdots, z_v; \lambda) \geqq 0.$$

COLUMBIA UNIVERSITY.

NOTE ON THE DERIVATIVE AND THE VARIATION OF A FUNCTION DEPENDING ON ALL THE VALUES OF ANOTHER FUNCTION.

BY PROFESSOR G. C. EVANS.

(Read before the American Mathematical Society, January 2, 1915.)

1. IN a recent article,* Fréchet has given a treatment of the differential of a function depending on a curve, by making use of and evaluating Riesz's expression of a linear relation in terms of a Stieltjes integral. According to Fréchet, if $F\overset{b}{\underset{a}{[\varphi]}}$ depends on all the values of $\varphi(x)$ between a and b, then

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* M. Fréchet, "Sur la notion de différentielle d'une fonction de ligne," *Transactions of the American Mathematical Society*, vol. 15 (1914), pp. 135–161.

if the differential of F exists it is given by the formula

$$dF[\varphi(x)] = \int_a^b \Delta\varphi(\xi)d\alpha_\varphi(\xi), \tag{1}$$

where $\Delta\varphi(\xi)$ is the increment in φ, and $\alpha_\varphi(\xi)$ is some function of finite variation. On the other hand, Volterra[*] has shown that, under certain conditions, the variation of F is given by the formula

$$\delta F[\varphi(x)] = \int_a^b F'[\varphi(x)|\xi]\delta\varphi(\xi)d\xi, \tag{2}$$

where $F'\varphi[(x) \mid \xi]$ is the functional derivative of F with respect to $\varphi(x)$ at the point ξ. The integral in (1), however, according to Fréchet, itself splits up into three parts, of which one has a form similar to (2), so that as a special case we should have

$$dF[\varphi(x)] = \int_a^b A[\varphi(x) \mid \xi]\Delta\varphi(\xi)d\xi. \tag{3}$$

It is the object of this short paper, in the first place, to derive the formula (2) under slightly less restrictive conditions than those of Volterra, and in the second place, by adopting a point of view more akin to that of Fréchet, to show the relation between equation (1) and equations (2) and (3).

2. We shall consider as a region for the argument $\varphi(x)$ that included between two given continuous functions $\Phi_1(x)$ and $\Phi_2(x)$, where $\Phi_1(x) < \Phi_2(x)$, in the interval $a \leqq x \leqq b$; i. e., the region

$$\Phi_1(x) \leqq \varphi(x) \leqq \Phi_2(x), \qquad a \leqq x \leqq b,$$

and we shall assume that $F[\varphi]$ is defined for every continuous function in that region, and is continuous.[†] This we shall call the assumption (α).

In addition to (α), in order to obtain formula (2), Volterra makes four assumptions I–IV. By a different method of proof, however,—the one which we first adopt—it is possible to arrive at (2) by means of (α), (II) and (III) alone, from which

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[*] V. Volterra, "Sopra le funzioni che dipendono da altre funzioni," *Rendiconti della R. Accademia dei Lincei*, vol. 3 (1887), pp. 97–105, 141–146, 153–158.

[†] We mean that $F[\varphi]$ has continuity of the zeroth order.

IV follows, but not (I) in its entirety.* These assumptions are as follows.

(II) Take an interval h within ab and give to $\varphi(x)$ within h a continuous variation $\theta(x)$, of one sign, such that $|\theta(x)| < \epsilon$. Denote by ΔF the corresponding change of F, and write $\sigma = \int_h \theta(x)dx$. It is assumed that the ratio $\Delta F/\sigma$ approaches a fixed limit (denoted by $F'[\varphi(x)|\xi]$ and spoken of as the functional derivative of F) as ϵ and h approach zero in an arbitrary manner, provided that the interval h always contains within itself the value $x = \xi$.†

(III) It is assumed that the ratio $\Delta F/\sigma$ approaches its limit uniformly with respect to all possible functions $\varphi(x)$ and values ξ.

Let us for convenience denote by (III$_1$) that part of (III) which requires uniformity with respect to the functional argument alone.

I. First Deduction of Formula (2).

3. The following theorem is introductory.

Theorem 1. *If for a certain continuous function $\varphi_0(x)$ the function $F[\varphi(x)]$, continuous in $\varphi(x)$, has a derivative $F'[\varphi(x)|\xi]$ for every value of ξ in a certain closed sub-interval $a'b'$ of ab,‡ that derivative remains finite and continuous throughout $a'b'$.*

To prove this theorem, let $\xi_1, \xi_2, \cdots$ be any infinite set of values of ξ in $a'b'$ having ξ_0 as a limiting point, and suppose that

$$F'[\varphi_0(x)|\xi_0] = t, \qquad \lim_{n=\infty} F'[\varphi_0(x)|\xi_n] = t'.$$

We shall assume that t' is finite; the case where t' is infinite occasions an obvious modification of the proof. Let $|t-t'| = p$, and suppose momentarily $p \neq 0$.

Give to $\varphi_0(x)$ a variation of one sign, $\theta_1(x)$, of the kind specified in (II), and take ϵ_1 and h_1 so small that we have the inequality

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* See § 4, and footnote.

† We understand here that $\theta(\xi) \neq 0$. This restriction turns out to be immaterial, but makes the definition correspond more closely to that of the ordinary derivative, where a restriction somewhat related to this is essential to the nature of the operation.

This definition requires obvious modification when $\xi = a$ or b and when $\varphi(x) = \Phi_1(x)$ or $\Phi_2(x)$.

‡ In particular, we may take $a' = a$ and $b' = b$.

$$(4) \qquad | F[\varphi_0(x) + \theta_1(x)] - F[(\varphi_0(x)] - t\sigma_1 | < \left|\frac{1}{2} p\sigma_1\right|.$$

We may, however, by taking a variation θ_2, small enough, and about a point ξ_n near enough to ξ_0, and adding to it a variation θ_3, small enough everywhere, yet different from zero at ξ_0, obtain a variation θ_1 for which is satisfied the inequality

$$(5) \qquad \begin{aligned} &| F[\varphi_0(x) + \theta_2(x) + \theta_3(x)] - F[\varphi_0(x)] \\ &\qquad\qquad - t'(\sigma_2 + \sigma_3) | < \left|\frac{1}{2} p(\sigma_2 + \sigma_3)\right|. \end{aligned}$$

For, since F is assumed to be continuous, the increment θ_3 may be made so small as to affect the difference $F[\varphi_0 + \theta_2] - F[\varphi_0]$ by as little as we please. But from (4) and (5), by taking $\theta_1 = \theta_2 + \theta_3$, it follows that

$$| (t - t')(\sigma_2 + \sigma_3) | < p \,| \sigma_2 + \sigma_3 |,$$

which is a contradiction. Hence $p = 0$, and the theorem is proved.

4. Let us now make the assumptions (α), (II), (III_1). It follows at once that *the derivative* $F'[\varphi(x) \mid \xi]$ *is continuous in regard to* $\varphi(x)$, *if* ξ *is any fixed value in the interval ab.* For we have

$$\left| F'[\varphi(x) \mid \xi] - \frac{F[\varphi(x) + \theta(x)] - F[\varphi(x)]}{\sigma} \right| < \eta,$$

and on account of the condition of uniformity,

$$\begin{aligned} &| F'[\varphi_2(x) \mid \xi] - F'[\varphi_1(x) \mid \xi] | \\ &\qquad \leqq | 1/\sigma \{F[\varphi_2(x) + \theta(x)] - F[\varphi_1(x) + \theta(x)]\} | + 2\eta. \end{aligned}$$

Hence first fixing the ϵ and h corresponding to $\theta(x)$ small enough so that 2η is less, say, than $\omega/2$, we can then take φ_2 near enough to φ_1, the $\theta(x)$ being fixed, so that the other part of the expression is also less than $\omega/2$. That is to say, by taking $| \varphi_2(x) - \varphi_1(x) |$ small enough, we can make the left hand member of the inequality as small as we please. Hence F' has continuity of the zeroth order with respect to φ.

From this it follows that F' is continuous uniformly with respect to φ, if φ is restricted to any family of curves whose ordinates are uniformly continuous functions of a finite number of parameters, over a perfect domain for those parameters, or if φ is restricted to a family of curves closed in the sense that the limiting curves are uniform limits; but not that F' is continuous in φ uniformly with respect to all continuous functions φ in the given region, nor even that F' is continuous in ξ uniformly for all continuous functions φ in the given region. *If however we make the whole of assumption* (III), it follows, as is easily verifiable, that *F' is continuous in ξ uniformly for all points ξ and all continuous curves φ in the region*, and hence that *F' is continuous in ξ and φ uniformly with respect to ξ and φ, provided that φ is restricted to a family of continuous functions, closed in the sense that the limiting functions are uniform limits.**

5. We can now prove, by means of the assumptions (α), (II), (III$_1$), something analogous to Rolle's theorem.

THEOREM 2. *Let $F[\varphi(x)]$ be a function for which* (α), (II), (III$_1$) *hold, and let $F[\varphi_1] = F[\varphi_2] = 0$, where $\varphi_1 - \varphi_2$ is a function which does not change sign in the interval ab, and is different from zero only in the interval $a'b'$. Then there is a function φ_0, of the pencil determined by φ_1 and φ_2, and a value $\xi_0 (a' \leqq \xi_0 \leqq b')$, such that $F'[\varphi_0(x) \mid \xi_0] = 0$.*

In fact, if we write $F[\varphi_1 + \omega(\varphi_2 - \varphi_1)]$ as a function of ω, $F(\omega)$, it will be continuous in ω, and for a certain value $\omega = \omega_0$ will attain its maximum or minimum. Let this value of ω determine the function φ_0, and for the sake of definiteness, let us assume that $F(\omega) - F(\omega_0)$ is not positive if ω is in the neighborhood of ω_0. For the sake of definiteness also, let us assume that the functional derivative of F is positive when $\xi = b'$. Then it must be positive throughout the whole of the closed interval $a'b'$, unless it vanishes at some point of that interval, since it is continuous in ξ. Let us assume that it does not vanish.

In order to show the falsity of this assumption, let us construct the functions

$$\begin{aligned} \psi_{\omega,t}(x) &= 0 && (a \leqq x \leqq t,\ b' \leqq x \leqq b) \\ (6) \qquad &= (x - t)\{\varphi_2(x) - \varphi_1(x)\} && (t \leqq x \leqq t + \omega) \\ &= \omega\{\varphi_2(x) - \varphi_1(x)\} && (t + \omega \leqq x \leqq b'), \end{aligned}$$

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* This is essentially what Volterra uses as his postulate (IV). See loc. cit., p. 99 and p. 101. His postulate (I) is that if $\theta(x)$ is any variation of φ in h (not necessarily of one sign) in absolute value less than ϵ, then $|\Delta F/\epsilon h| < M$, uniformly.

and consider for what value of t, $t = \bar{t}$, the difference

$$F[\varphi_0(x) + \psi_{\omega,\, t}(x)] - F[\varphi_0(x)]$$

attains its upper limit, if ω is kept constant. We know that there will be such a value, since F is a continuous function of t, and we see directly that we can find ω' small enough so that if $\omega \leqq \omega'$ that value must be greater than a'. For if not, on account of the uniformity of the condition (III$_1$) and the resulting continuity of the derivative with regard to its functional argument, it would follow that we might take ϵ and h about a' so small that

$$F[\varphi(x) + \theta(x)] - F[\varphi(x)] > 0,$$

where $\theta(x)$ is any variation in the interval h, everywhere in absolute value less than ϵ,[*] and $\varphi(x)$ is any one of the functions $\varphi(x) = \varphi_0(x) + \psi_{\omega,\, t}(x)$. But this means that we could take ω small enough so that we should have $F[\varphi_0 + \omega(\varphi_2 - \varphi_1)] - F[\varphi_0] > 0$, which would be contrary to hypothesis.

Let us now take a series of values ω_n which approach zero as a limit. The corresponding series of values t_n of $\bar{t}$ has at least one limiting value, and any one of these limiting values, which may in particular be the point a', we may take as our ξ_0. This gives us our contradiction. For if $F'[\varphi_0(x) \mid \xi_0]$ were not zero, we could, owing to the uniformity of the condition (III$_1$), construct a function $\psi_{\omega_n,\, t}(x)$, with $t < t_n$, for which we should have

$$F[\varphi + \psi_{\omega_n,\, t}] > F[\varphi + \psi_{\omega_n,\, t_n}].$$

Our theorem is therefore proved.

6. The law of the mean is a consequence of this theorem in the same way as in differential calculus it is a consequence of Rolle's theorem.

LAW OF THE MEAN.[†] *Let* $F[\varphi(x)]$ *be a function for which* (α), (II), (III$_1$) *hold, and let* φ_1 *and* φ_2 *be two continuous functions in the given region, such that* $\varphi_1 - \varphi_2$ *does not change sign in the interval ab, and is different from zero only in the interval a′b′.*

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[*] If $a' = a$, it is taken for granted that $\varphi_2(a) - \varphi_1(a)$ and $\theta(a)$ need not necessarily vanish. Also if $b' = b$, it is assumed that $\varphi_2(b) - \varphi_1(b)$ need not necessarily vanish.

[†] This theorem is due to Volterra (loc. cit., p. 103) who establishes it by means of formula (2), and thus as a consequence of the hypotheses (α) (I)–(IV).

Then there is a function φ_0 *of the pencil determined by* φ_1 *and* φ_2, *and a value* ξ_0 $(a' \leqq \xi_0 \leqq b')$ *such that*

$$F[\varphi_2(x)] - F[\varphi_1(x)] = F'[\varphi_0(x) \mid \xi_0] \int_a^b \{\varphi_2(x) - \varphi_1(x)\} dx. \tag{7}$$

7. We may now proceed, by means of these theorems and the hypotheses (α), (II), (III), to establish the formula (2). Let us consider first a continuous function $\psi(x)$ which does not change sign in the interval ab, and form the function $F(\omega) = F[\varphi_0 + \omega\psi]$. We shall endeavor to calculate $dF/d\omega$ for $\omega = 0$.

If we divide up the interval ab into parts $a_i a_{i+1}$, every one of which is in magnitude less than δ, we can write

$$\begin{aligned} F[\varphi_0 + \omega\psi] - F[\varphi_0] = \sum_{i=0}^{n-1} F[\varphi_0 + \psi_{\omega, a_i}] - F[\varphi_0 + \psi_{\omega, a_{i+1}}] \\ + F[\varphi_0 + \omega\psi] - F[\varphi_0 + \omega\psi_{\omega, a}], \end{aligned}$$

where $a_0 = a$ and $a_n = b$. But this is the same as the expression

$$\begin{aligned} \omega \sum_{i=0}^{n-1} F'[\varphi_0 + \psi_{\omega_i, a_{i+1}} + \gamma_i(\psi_{\omega, a_i} - \psi_{\omega_i, a_{i+1}}) \mid \xi_i] \\ \times \int_a^b (\psi_{\omega, a_i} - \psi_{\omega, a_{i+1}}) dx + F[\varphi_0 + \omega\psi] - F[\varphi_0 + \psi_{\omega, a}], \end{aligned}$$

where $|\gamma_i| < 1$ and $a_i \leqq \xi_i \leqq a_{i+1}$. We may for our purposes take all the intervals equal, and also $\delta = k\omega$ where k is any fixed number greater than 2. If then we take the limit of $\{F[\varphi_0 + \omega\psi] - F[\varphi_0]\}/\omega$ as ω approaches zero, we verify from the uniformity of assumption (III) and the results of § 4 that $dF/d\omega$ exists for $\omega = 0$, and is given by the formula

$$\left(\frac{dF}{d\omega}\right)_{\omega=0} = \int_a^b F'[\varphi(x) \mid \xi]\psi(\xi) d\xi. \tag{8}$$

Hence it follows that

$$\left(\frac{dF}{d\omega}\right)_{\omega=\omega} = \int_a^b F'[\varphi(x) + \omega\psi(x)]\psi(\xi) d\xi. \tag{9}$$

8. Consider now the general case where $\psi(x)$ is an arbitrary continuous function. We can write it as $\psi(x) = \psi_1(x) + \psi_2(x)$,

where

$$\psi_1(x) = \tfrac{1}{2}\{\psi(x) + |\psi(x)|\},$$
$$\psi_2(x) = \tfrac{1}{2}\{\psi(x) - |\psi(x)|\},$$

functions which do not change sign. Moreover if we consider the function $F[\varphi_0 + \omega_1\psi_1 + \omega_2\psi_2]$ we have the formulas

$$\frac{\partial F}{\partial \omega_1} = \int_a^b F'[\varphi_0 + \omega_1\psi_1 + \omega_2\psi_2 \mid \xi]\psi_1(\xi)d\xi,$$
$$\frac{\partial F}{\partial \omega_2} = \int_a^b F'[\varphi_0 + \omega_1\psi_1 + \omega_2\psi_2 \mid \xi]\psi_2(\xi)d\xi.$$

On account of the uniform continuity of F' over ξ and the two-parameter family of curves defined by ω_1 and ω_2, it follows that $\partial F/\partial\omega_1$ and $\partial F/\partial\omega_2$ are continuous functions of ω_1 and ω_2. Hence if we put $\omega_1 = \omega_2 = \omega$, we find that $dF/d\omega$ exists and is given by

$$\frac{dF}{d\omega} = \left(\frac{\partial F}{\partial \omega_1}\right)_{\substack{\omega_1=\omega\\ \omega_2=\omega}} + \left(\frac{\partial F}{\partial \omega_2}\right)_{\substack{\omega_1=\omega\\ \omega_2=\omega}},$$

which reduces to the form (9), since $\psi_1 + \psi_2 = \psi$. For $\omega = 0$, equation (9) reduces to equation (8), which holds for any continuous function $\psi(x)$, and is equivalent to (2).

From (9) we obtain (with Volterra) another law of the mean in the form

$$(10)\quad F[\varphi(x) + \psi(x)] - F[\varphi(x)] = \int_a^b F'[\varphi(x) + \theta\psi(x) \mid \xi]\psi(\xi)d\xi,$$

where $0 < \theta < 1$.

II. OTHER METHODS OF DEDUCING FORMULA (2).

9. We may obtain the formula (8) and hence the formula (2) from more general points of view. One point of view, which we shall consider, is closely related to the ideas which have just been developed; a second is a modification of the procedure of Fréchet.

For simplicity let us replace (α) by a new hypothesis (α'), the difference being that F is defined not only for all continuous functions in the given domain, but also for all functions in the given domain which have merely a finite number of discontinuities.*

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* It is sufficient for what follows if we admit merely discontinuities *of the first kind*, so called (Lebesgue).

Let $\psi(x)$ be a continuous function, *of one sign*, and construct the functions ${}_\alpha\psi_\beta(x)$ defined as follows:

$$\begin{aligned} {}_\alpha\psi_\beta(x) &= 0 \quad (a \leqq x < \alpha,\ \beta < x \leqq b) \\ &= \psi(x) \qquad (\alpha \leqq x \leqq \beta). \end{aligned}$$

Let us consider besides (α'), the following postulates (β), (γ):

$$(\beta) \qquad \lim_{\omega=0,\, \alpha=a',\, \beta=b'} \frac{\Delta F}{\sigma} \text{ exists} \quad (a \leqq a' \leqq b' \leqq b),$$

where

$$\Delta F = F[\varphi(x) + \omega_\alpha\psi_\beta(x)] - F[\varphi(x)]$$

$$\sigma = \omega \int_\alpha^\beta {}_\alpha\psi_\beta(x)dx.$$

This limit will depend on $\varphi(x)$, $\psi(x)$, a' and b'; let us call it $G[\varphi(x), \psi(x) \mid a', b']$.

(γ) $\quad G[\varphi(x), \psi(x) \mid a', b']$ is continuous in $\varphi(x)$.

10. If (α') and (β) are satisfied, we can verify directly that when a' and b' are equal, the function G is independent of $\psi(x)$, and we can therefore write

$$G[\varphi(x), \psi(x) \mid a', a'] = G[\varphi(x) \mid a'].$$

If F happens to have a functional derivative F', then

$$(11) \qquad G[\varphi(x) \mid a'] = F'[\varphi(x) \mid a'].$$

If (α') and (β) are satisfied, we see also that if we let $|\alpha - a'|$, $|b' - \beta|$ approach zero with ω as functions of ω, then

$$\left(\frac{d}{d\omega}F[\varphi(x) + \omega\, {}_\alpha\psi_\beta(x)]\right)_{\omega=0} = G[\varphi(x), \psi(x) \mid a', b'] \int_{a'}^{b'} \psi(x)dx \quad \text{if } a' \neq b', \text{ and}$$

$$= 0 \quad \text{if } a' = b'.$$

11. If (α'), (β), (γ) are satisfied, we can deduce, by making use of partial derivatives as in § 8, that

$$(13) \qquad \begin{aligned} &G[\varphi(x), \psi(x) \mid a', b'] \int_{a'}^{b'} \psi(x)dx + G[\varphi(x), \psi(x) \mid b', c'] \\ &\times \int_{b'}^{c'} \psi(x)dx = G[\varphi(x), \psi(x) \mid a', c'] \int_{a'}^{c'} \psi(x)dx. \end{aligned}$$

And from this and the definition of $G[\varphi(x) \mid a']$ it follows that $G[\varphi(x), \psi(x) \mid a', b']$ is a continuous function of its two arguments a', b' in the region $a \leqq a' \leqq b' \leqq b$, and is hence uniformly continuous with respect to them. But these properties lead us to a formula equivalent to (8). For we have

$$\left(\frac{d}{d\omega}F[\varphi + \omega\psi]\right)_{\omega=0} = G[\varphi(x), \psi(x) \mid a, b] \int_a^b \psi(x)dx,$$

and if we split the right-hand member into n parts, according to (13), and take the limit as n becomes infinite, remembering that G is uniformly continuous in a', b', we obtain the result

$$(14) \qquad \left(\frac{d}{d\omega}F[\varphi + \omega\psi]\right)_{\omega=0} = \int_a^b G[\varphi(x) \mid \xi]\psi(\xi)d\xi,$$

where $\psi(x)$ is any continuous function which does not change sign. This again is generalized to hold for any continuous function $\psi(x)$, by the method of § 8.* Therefore, *formula* (14), *where* $\psi(x)$ *is any function continuous* $a \leqq x \leqq b$, *is a consequence of postulates* (α'), (β), (γ).†

12. It is now evident what sort of restriction is sufficient in order that formula (1) may reduce to (2). In fact we can deduce (14) from (α') and the two following hypotheses:

(β') Let $\psi(x)$ be any limited function, continuous except for a finite number of discontinuities (of the first kind); then we assume that

$$\left(\frac{d}{d\omega}F[\varphi + \omega\psi]\right)_{\omega=0}$$

exists and is distributive with respect to ψ, i. e.,—

$$\left(\frac{d}{d\omega}F[\varphi + \omega(\psi_1 + \psi_2)]\right)_{\omega=0} = \left(\frac{d}{d\omega}F[\varphi + \omega\psi_1]\right)_{\omega=0} + \left(\frac{d}{d\omega}F[\varphi + \omega\psi_2]\right)_{\omega=0};$$

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[*] If in the definition of $G[\varphi(x), \psi(x) \mid a', b']$ we do not allow ω to change sign as it approaches zero, we are led to two functions $G_+[\varphi(x) \mid \xi]$ and $G_-[\varphi(x) \mid \xi]$, whence, instead of (14) for the general ψ, we have

$$(14') \quad \left(\frac{d}{d\omega}F[\varphi + \omega\psi]\right)_{\omega=0} = \frac{1}{2}\int_a^b \{G_+[\varphi(x) \mid \xi](\psi(\xi) + \mid \psi(\xi) \mid) + G_-[\varphi(x) \mid \xi](\psi(\xi) - \mid \psi(\xi) \mid)\}d\xi.$$

[†] If $\varphi(x) = \Phi_1(x)$, $\psi(x)$ cannot be negative; if $\varphi(x) = \Phi_2(x)$, $\psi(x)$ cannot be positive.

(γ') The function $G[\varphi(x) \mid \xi]$ defined in (β) exists for $a \leqq \xi \leqq b$.

The postulate (β') is substantially the hypothesis used by Fréchet with (α) in defining the existence of a differential.* Moreover if we restrict ourselves to a function $\bar{\psi}(x)$ of one sign, we see that the function

$$G'[\varphi(x), \psi(x) \mid a', b'] = \lim_{\omega=0} \frac{F[\varphi(x) + \omega_{a'}\psi_{b'}(x)] - F[\varphi(x)]}{\omega \int_{a'}^{b'} \psi(x)dx}$$

exists, provided that $a' \neq b'$, and from (β') we see that the relation (13) holds with G' for G. And if we define

$$G'[\varphi(x), \psi(x) \mid a'a'] = G[\varphi(x), a'],$$

it follows as in (11) that G' is uniformly continuous in a' and b' in the region $a \leqq a' \leqq b' \leqq b$. Hence we obtain again formula (14) for a function ψ of one sign, and, finally, for any function ψ if ψ is limited and has merely a finite number of discontinuities (of the first kind).

13. The advantage of the last set of assumptions (α'), (β'), (γ') is that they do not demand the continuity of $G[\varphi(x) \mid \xi]$, or $F'[\varphi(x) \mid \xi]$ if it exists, with respect to the functional argument $\varphi(x)$. If for a given $\varphi(x)$, we suppose that $G[\varphi(x) \mid \xi]$ does not exist for certain special values of ξ, we have the case that F depends in a special manner on $\varphi(x)$ for those values of x.† The case where $A[\varphi(x) \mid \xi]$ in (3) is discontinuous comes under this specification. The property of possessing a functional derivative in general, however, seems to depend on fundamental properties of continuity with respect to aggregates of functions, the study of which is thereby rendered specially inviting.

THE RICE INSTITUTE,
February, 1915.

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* M. Fréchet, loc. cit., p. 141. In order to justify the substitution of a postulate akin to (α') rather than (α) see F. Riesz, "Les opérations fonctionelles linéaires," *Annales scientifiques de l'Ecole Normale Supérieure*, vol. 31 (1914), p. 2, who shows that a linear relation such as that expressed in the conception of differential can always be extended by definition to apply to certain classes of discontinuous arguments.

† V. Volterra, loc. cit., p. 144.

NON-EUCLIDEAN GEOMETRY.

The Elements of Non-Euclidean Geometry. By D. M. Y. SOMMERVILLE. London, G. Bell and Sons, 1914. 16mo. xvi+274 pp.

FEW recent writers upon non-euclidean geometry have approached their task with better chances of success than attended Dr. Sommerville in the preparation of the present volume. Anyone who has seen his scholarly and painstaking "Bibliography of Non-Euclidean Geometry"* will realize that in so far as a knowledge of what others have written upon a subject is a desirable qualification, the present author was most fortunately placed. Furthermore he is the happy possessor of an excellent literary style. A book written by such a writer should be interesting and stimulating; the present book has both of these characteristics. The choice of material is admirable, and the narrative continually illumined by historical notes.

When the fairies were invited to the christening of the Sleeping Beauty one of the sisterhood was unfortunately overlooked, and her absence caused all the trouble that came afterwards. So here, one thing is lacking, singleness of aim. Says the author (page vii):

"It is hoped that the book will prove useful to the scholarship candidate in our secondary schools who wishes to widen his geometrical horizon, to the honours student at our universities who chooses geometry as his special subject, and to the teacher of geometry in general who desires to see how far strict logical rigour is made compatible with a treatment of the subject matter capable of comprehension by schoolboys."

Does not this programme spell "failure" from the start? Complete rigor and a treatment comprehensible by schoolboys, even by Scotch ones, who indubitably work harder and know more than Americans of like age, are so far incompatible that it is quite useless to make the attempt. The needs of the schoolboy and of the candidate for honors are so different that a book intended for both will suit neither. In the present work if we confine ourselves to the first four chapters, which the author declares (page vi) constitute the rudiments of the subject, and omit all material in fine print, we still find such topics as the exponential function with a complex argument, hyperbolic functions, imaginary points and lines, the differential equation

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* London, Harrison, 1911. Reviewed in the BULLETIN, vol. 18, Feb., 1912.

$$\frac{d^2a}{dr^2}+\frac{a}{k^2}=0,$$

the bizarre integral

$$\int_0^c k\sin\frac{c}{k}\,dAdc,$$

the plane at infinity, and the desmic configuration. We have yet to find any actual schoolboy who was able to make very much of such expressions. But if we consider the book from the point of view of the "honours student" we observe that he will find a romantic optimism in many statements and proofs altogether at variance with what he is learning in his analysis and his algebra. The author's disclaimer (page vi) of any attempt to make the book rigorously logical with a detailed examination of all assumptions is no sufficient excuse.

These are surely serious charges to bring against any author, especially against one so well equipped as is Dr. Sommerville. Let us try to sustain them in detail, without losing sight of the various attractive features which the book surely possesses.

The first chapter is purely historical and deals with the landmarks in the history of non-euclidean geometry, much as they are described in a score of books. Historical or not, the author is true to his British didactic instinct and closes this chapter, like all the others, with a number of examples for the student to work out. Surely the British text-book writers lead the world in this respect. Says Cremona in the preface to the English edition of his Geometria proiettiva:*

"Unless I am mistaken the preference given to my Elements over the many treatises on modern geometry published on the continent is to be attributed to the circumstance that in it I have striven, to the best of my ability, to imitate the English models. . . . I aimed therefore at simplicity and clearness of expression, and I was careful to supply an abundance of examples of a kind suitable to encourage the beginner."

If previous writers of text-books on non-euclidean geometry have omitted such examples, was it from conviction or laziness?

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* Second edition, Oxford, 1893, p. xiii.

In Chapter II we begin the systematic treatment of the hyperbolic plane. The writer says little about axioms, except to give Hilbert's classification into axioms of connection, order, congruence, continuity, and parallelism. It is by no means easy to say how the author wants us to treat these axioms. The inference is that we are to accept all save the last, for we read (page 27): "We shall assume as deductions from them the theorems relating to the comparison and addition of segments and angles," and two pages later we have Pasch's axiom that a line which meets a side of a triangle and a second side produced meets the third side. But this axiom depends for its statement upon the axioms of order, i. e., the axioms of an open order. Yet if we accept an open order at the outset why trouble ourselves at all about the elliptic plane where a straight line has a closed order and where Pasch's axiom may not be true? The fact is that the whole axiom question is beset with difficulties. If a writer who has not had the needful special training undertake to make up his own set of axioms, he is likely to make a botch of it; if he accept uncritically a set which some one else has developed, he is in grave danger of running into contradictions.

"Revenons à nos moutons." The first dozen pages of Chapter II go to a discussion of parallel lines in the hyperbolic plane and give the important theorems in good form. All is clear and well defined. The first break occurs on page 39 where we read:

"*Two parallel lines can therefore be regarded as meeting at infinity, and further the angle of intersection must be considered as being equal to zero.*"

We find further on (page 46):

"We shall extend the class of points by including a class of fictitious points called points *at infinity*. These points function in exactly the same way as ordinary, or, as we shall say, actual points. . . . On every line there are two points at infinity."

Our comment on these statements is as follows: We only know two ways of extending the class of points to include new members. We may define the new points by means of already recognized figures, as for instance, we might define a "point at infinity" as the totality of lines parallel to a given line and to one another, or, secondly, we might define a "point at infinity" by a set of postulates as

There exists a class of P.I.'s.

Each line contains two P.I.'s.
Each point and each P.I. determine a line.
Every two P.I.'s determine a line.

Starting with either of these methods we may go on to define lines and planes at infinity, and then similarly ultra-infinite points, lines and planes. Either plan is permissible, neither is entirely simple. But when our present author tells us (page 48) that a bundle of lines perpendicular to a plane *have* an ideal vertex, and further that "*ideal points* thus introduced behave exactly like actual points" we are left wondering. They surely do if we confine ourselves to projective properties, but if, perchance, we seek the distance from an actual to an ideal point we are in very serious difficulty.

There are two other points to be noticed while we are upon these thorny pages of the book. We read (page 41):

"Thus the distance between the two given lines AB' and LX first diminishes and then tends to infinity. It must therefore have a minimum value." This is, of course, a pure assumption and should either be proved, or made explicitly. Then we read in the note to page 46:

"The definition of a *conic* which we shall use is 'a plane curve which is cut by any straight line in its plane in two points.' For the explanation of the case of 'imaginary intersections' see Chapter III, § 5."

Here, if we overlook the removable objection that a tangent meets a conic in but one point, we still wonder what is the author's definition of a curve. If he mean an analytic curve, the definition for a conic is entirely proper. For if we take the origin upon such a curve and the axes parallel to the asymptotes, the abscissa and ordinate of every other point will be analytic functions of the slope of the line from the point to the origin, and these functions are single valued, and have single valued inverses. Hence we may express the curve in the form

$$x = \frac{al + b}{cl + d}, \quad y = \frac{a'l + b'}{c'l + d'}, \quad l = \frac{y}{x},$$

and we have the usual conic of commerce. But in the present book there is, up to the present point, no machinery for an exact statement of this sort and as for the method of introducing imaginaries, well, we shall deal with that presently.

The last part of the chapter goes to the development of trigonometry following the classical methods of Lobachevsky which depend upon the parallel angle. Here everything is elementary, though at times a bit involved. Moreover, it is stated upon page 58 that the solution of

$$f(x + y) = f(x)f(y)$$

must be $f(x) = c^x$. This is true if, and only if, the restriction be imposed that $f(x)$ shall be a continuous function not $= 0$.* Our author can not have been ignorant of the necessity for this restriction, since on page 114 in discussing the corresponding equation for $\cos x$ he mentions the requirement of continuity. Query, how many schoolboys will understand what a functional equation means anyway? The chapter ends with an ingenious discussion of the relation between the defect of a triangle and its area.

Chapter III is given to elliptic geometry. This must be worked out "ab initio" since the methods of hyperbolic geometry, based on the parallel angle, are inapplicable. A curious omission occurs (page 89) where the author recognizes that if a straight line be a closed curve, two coplanar lines might intersect once or twice, but fails to show that they might not have more than two intersections. As a matter of fact the proof is a bit delicate. He rightly appreciates that the most attractive feature of elliptic geometry is the perfect duality between point and plane, angle and distance. To bring this out he needs projective geometry, and since he deems it unwise to assume a knowledge of the latter on the part of his readers he gives a short introductory account of the subject on pages 93–98. This is no easy task. The idea of a cross ratio is fundamental in projective geometry, but the elementary definition in terms of the ratios of certain segments is invalid in non-euclidean space, while (wrongly, as we believe) he considers that a definition based upon successive quadrangle and quadrilateral constructions is too difficult or abstract. He therefore adopts the following unhappy expedient. We read (page 94):

"If two ranges of points are made to correspond in such a way that to every point P on one range corresponds uniquely a point P' on the other and *vice versa*, the ranges are said to be *homographic*," and on page 95: "It can be proved that two homographic ranges can always be connected by a finite number of projections and intersections." This familiar statement has always reminded us of the sea-serpent. It is based upon an illusion, but is of venerable antiquity, is believed in by many, and is indestructible. Yet there is no difficulty in showing its falsity. Let us ask the writer in what domain we are supposed to be. If the answer be the real domain, since nothing yet has been said about imaginaries, we ask him how many projections and intersections are necessary to accomplish the transformation

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* Conf. Wilson, "Note on the function satisfying $\phi(u)\phi(v) = \phi(u + v)$" *Annals of Mathematics*, series 2, vol. 1, 1899, and Darboux, "Sur le théorème fondamental de la géométrie projective," *Math. Annalen*, vol. 17 (1880).

$$x' = x^3.$$

If, however, we are in the complex domain, we have the classical example

$$x' = \bar{x},$$

i. e., each point of a line corresponds to its conjugate imaginary. Of course the statement is true if the domain be complex, and the transformation analytic, but these are just the restrictions which are usually overlooked.

The imaginary first appears explicitly on page 97 where we read, in discussing involutions: "If two real self-corresponding points do not exist we introduce by definition conjugate pairs of 'imaginary' points, much in the same way that ideal points were introduced in hyperbolic geometry." With this epigrammatic statement the author leaves the matter; he now feels just as free to use imaginary points as real ones.

It is fair to say that this happy solution of the imaginary difficulty is not new. The principle, so far as we understand it, is this. If by a continuous change we can make two real points coalesce and then disappear we are free to say that they have been replaced by two imaginary points, and repel with scorn all inquiries as to the nature of these latter. Let us make a careful drawing of the curve.

$$(x^2 + y^2 + 1)^{1492}(y - x^3) + \epsilon = 0,$$

where ϵ is microscopic. This curve will lie extremely near to

$$y = x^3.$$

Some lines will meet our curve in three real points, others in only one. Shall we, therefore, say that the latter meet the curve in one real and *two* imaginary points? If our pupils be too young to comprehend the truth in these matters, let us, at least, tell them nothing else.

A slight slip of a different sort occurs on page 99, where we read "Similarly, in three dimensions a polar system determines a surface of the second degree, or quadric surface." The author has clearly forgotten the null system. The remainder of the chapter is devoted to paratactic lines, Clifford surfaces, and trigonometric relations. The subject is interesting and the treatment good, although we find on page 111 the erroneous statement "*all the common tranversals of two right paratactic lines are left paratactic lines.*"

The fourth chapter is given to analytic geometry. A point in the plane is determined by its Weierstrass coordinates, namely

$$x = k \sin\frac{r}{k}\cos\theta, \quad y = k\sin\frac{r}{k}\sin\theta, \quad z = \cos\frac{r}{k},$$

where r is the distance from the origin, θ is the angle between the radius vector and the positive half of the X axis, and k the space constant. The subject matter is beset with fewer pitfalls than that which preceded, and the handling is satisfactory. The chapter closes with the account of two configurations, the triangle configuration of Desargues and the desmic tetrahedron configuration of Stéphanos.

A distinct change comes over the book with Chapter V. This excellent chapter is given to various representations of non-euclidean space in euclidean space. Three such representations are discussed. First we have the projective representation, and Cayley's projective metric; secondly the geodesic representation where the non-euclidean plane is developed on a euclidean surface of constant curvature, and lastly the conformal representation where non-euclidean lines appear as euclidean circles orthogonal to a fixed circle. Our only criticism is the mild wish that the chapter had been a little longer, and that the author had pointed out more clearly that in some cases we are able only to develop a part of our space.

Our author has next the curious idea of interpolating another philosophical chapter, to bring the history of non-euclidean geometry to date, and to discuss an interesting philosophical question. These are rather deep subjects to be disposed of in a twenty page chapter, and the treatment of the modern abstract view of geometry is scant. The particular philosophical question is this: is our space of experience euclidean or non-euclidean? The author asks (page 203):

(1) Can the question of the true geometry be settled a posteriori or experimentally?

(2) Can it be decided on philosophical grounds?

(3) Is it, after all, a proper question to ask, one to which an answer can be expected?

Taking these up in turn he first shows that if the space constant be not infinite, it is so large that we shall never make any serious error in assuming that our space is euclidean. The second question is easily disposed of: no one believes nowadays that we can settle the question of parallels a priori. The third idea, that the question is improper, and that we can no more say that the parallel axiom is true or false, than that the metric system is true or false has the great authority of Poincaré, and we believe that Dr. Sommerville inclines to a like opinion. He puts the case as follows (page 209):

"A further point—and this is the 'vicious circle' of which we spoke above—arises in connection with the astronomical attempts to determine the nature of space. These experiments are based upon the received laws of astronomy and optics, which are themselves based upon the euclidean assumption. It might well happen then, that a discrepancy observed in the sum of the angles of a triangle could admit of an explanation by some modification of these laws, or that even the absence of any such discrepancy might still be compatible with the assumptions of non-euclidean geometry."

This is rather a fascinating idea. It is evident, for instance, that if light were not propagated along straight lines, it would be hard to determine the parallel question astronomically. However, leaving aside the minor doubt as to just how far our science of physical optics *is* dependent upon the parallel postulate, the root of the difficulty seems to us that the argument is a little too strong. Could we not render all scientific deduction impossible by similar considerations? If we make any experiment whatever, the results can be interpreted in a transfinite number of ways and the best that we can hope for is to find what conclusions are compatible with various accepted hypotheses. Let us take a simple question. Are there any mountains on the other side of the moon? At present it seems hard to find a crucial experiment to settle the matter. Suppose, however, that in the course of centuries we set up some sort of wireless telegraphy with those eminent engineers who superintend the construction of canals on Mars, and put the question to them. We may well imagine their replying " We do indeed observe there markings which are in every way similar to those markings on the earth to which you give the name of Himalayas." What conclusion should we then draw? It might be that the laws of optics for the Martians were different from our own, it might be that whereas rays of light reflected from the earth travelled in straight lines those reflected from the other side of the moon pursued spirals. Endless other conjectures are possible. The obvious conclusion is, however, that there are such mountains, and that no vicious circle was involved in asking the question.

The seventh chapter is devoted to circles, and in fact, the amount of attention paid to circles and circle transformations is a distinguishing characteristic of the book. The treatment is greatly clarified by the use of a transformation first given analytically by Hausdorf* and later synthetically by Liebmann.† Confining our attention to the real and actual domain of the hyperbolic plane, we take a ground plane F. Then if E be any other actual plane, its intersection with the absolute quadric will be orthogonally projected upon F into a circle, if we include under this head not only circles with actual centers, but horocycles and equidistant curves. Conversely, every such curve can be reached in this way. This being shown, our author says (page 230):

" Hence there is a (1, 1) correspondence between the circles in a plane and the planes in space."

" Ahimè! " before saying this why did he not try to run his machine backwards? Had he done so, instead of taking Liebmann's dictum, he would have perceived that two planes which lie symmetrically with regard to F give the same circle therein, so that the correspondence is not (1, 1) but (1, 2). Hausdorf, proceeding analytically, points this out with care‡ and subsequently changes it to a (1, 1) correspondence by covering the plane over with two different layers of points.

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* "Analytische Beitrage zur nicht-euklidische Geometrie," *Leipziger Berichte*, vol. 51 (1899).

† "Synthetische Ableitung der Kreisverwandtschaften, etc.," ibid., vol. 54 (1902).

‡ Loc. cit., p. 177.

It is curious that this mistake was not brought to our author's notice by the following inconsistency. He points out (page 218) that two circles may intersect in four (real) points, the resulting angles being equal in pairs, but warns us at the top of the next page that the four angles are not, in general, equal. We then reach this interesting theorem (page 231):

The angle between two planes is equal to the angle of intersection of their marginal images.

The marginal image of a plane is the circle which corresponds in the present scheme. Of course the correct theorem is this:

Given two circles in F, the one represented by the planes E_1E_2, *the other by* G_1G_2. *Two angles of these circles will be equal to the angle of the planes* E_1G_1 *(or* E_2G_2*), the other two will be equal to the angle of the planes* E_1G_2 *(or* E_2G_1*).*

Chapter VIII discusses circular transformations both geometrically and analytically. Part of the analytic work seems a bit messy; the geometrical treatment is ingenious, especially the proofs of the theorems that every circular transformation of the non-euclidean plane is conformal, that it can be factored into the product of a congruent transformation and an inversion or radiation, and that every circular transformation of the plane corresponds to a congruent transformation of space. These are valuable theorems which we are glad to see in such a work. Unfortunately we must note one more oversight, analogous to those which marred the chapter preceding, and common to most writers upon this subject.*

Let us take what our author calls (page 241) a *hyperbolic inversion*. Here corresponding points P and P' are collinear with the center of inversion O, the product of the hyperbolic tangents of the halves of their distances from O is constant, and they *are on the same side of* O. The analogy to euclidean inversion seems complete, for here we may define inverse points as being collinear with O, the product of their distances therefrom is constant, and *they are on the same side of* O. There is, however, an important distinction. The requirement that the two points should be *on the same side of* O is not an analytic requirement. In the euclidean plane it is unessential, for we may also define inverse points as being collinear with O and conjugate with respect to the circle of inversion. In the non-euclidean plane we have not this refuge, and, analytically speaking, a point in general position has *two inverses*. We bring out this important fact as follows.

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* We make an exception in favor of a really careful article by Beck, "Ein Seitenstück zur Mobiusschen Theorie der Kreisverwandschaften," *Transactions Amer. Math. Soc.*, vol. 11 (1910). Apparently Dr. Sommerville was not familiar with this article, at any rate there is no mention of it in his bibliography. Beck's circles are oriented, and treated as the envelops of oriented straight lines. His paper might equally well have been called a Seitenstuck to Laguerre's "Géométrie de direction."

We may map the upper half of the euclidean plane ($y > O$) upon the actual domain of the hyperbolic plane in such a way that the correspondence is one to one, conformal, and carries circles to circles; i. e., a circle in the hyperbolic plane will correspond either to a circle in the euclidean plane, or to a part of such circle and the reflection of the rest in the x axis. Let P be a point of the upper euclidean half-plane and c_1 any circle entirely in that half-plane. Let P' and c_1' be the corresponding point and circle of the hyperbolic plane. Then the inverses of P' in c_1' will be the points which correspond to the inverse of P in c_1 and that which corresponds to the reflection in the x axis of the inverse of P in the reflection of c_1. We may go a step further in this direction, and announce a theorem which is, perhaps, new.

The only analytic circular transformations of the non-euclidean plane which are single valued are congruent transformations.

It is ridiculously easy to prove this analytically, but perhaps it will be more sportsmanlike to follow the methods of the book before us. Since our transformation carries a point to a point, a circle to a circle, and the points of a circle to points of a circle for our plane F, it is (page 237) a conformal transformation. Hence a circle orthogonal to itself will go into another such, i. e., a null circle into a null circle. In space we have a transformation of plane to plane, which carries a plane tangent to the absolute into another such plane, in fact a congruent transformation of space (page 238). Since the transformation of F is to be one-to-one, two planes tangent to the absolute which correspond to the same null circle in F, i. e., two planes tangent to the absolute whose common line is in F, will correspond to two other such planes. Here a line in F is carried into a line in F, i. e., our congruent transformation carries F into itself.

The concluding chapter goes to a classification of non-degenerate conics under the non-euclidean congruent group. The subject matter is not particularly novel, and the treatment calls for no special comment.

If we were to pass a final judgment upon the book before us it would be this: It is in some respects an improvement upon its predecessors; it is by no means the best book that the present author could have written.

J. L. COOLIDGE.

CAMBRIDGE, MASS.,
December, 1914.

MINKOWSKI'S WORKS.

Gesammelte Abhandlungen von Hermann Minkowski. Herausgegeben von D. HILBERT. Leipzig, B. G. Teubner, 1911. Band 1, xxxi + 371 pp. Band 2, iv + 465 pp. Two portraits.

MINKOWSKI's work divides itself naturally, and his collected works are divided, into four parts: Theory of quadratic forms, 242 pages, Geometry of numbers, 230 pages, Geometry, 180 pages, Physics, 163 pages. In addition to this the volumes before us contain the author's address on Dirichlet, 15 pages, and Hilbert's commemorative address on Minkowski, 27 pages. This heartful and touching tribute of a life-long friend and fellow-worker is in reality also a critical review of Minkowski's great achievements in mathematical science, and it may be that the best thing for us to do in reviewing these volumes would be to follow the example of the reviewer in another Bulletin* and translate the chief portions of that address. The availability of the address in the original, where it should be read as a whole, and, in abstracts, in French makes repetition here seem really unnecessary.

We are accustomed to precocious exhibitions of genius in mathematicians, and we often cite the case of Galois, who died in his twenty-first year after accomplishing work of which the fundamental importance was not and perhaps could not be appreciated until a much later date. Minkowski in his eighteenth year submitted to the Paris Academy a memoir on quadratic forms with integral coefficients which fills 142 pages of his collected works and which received the Grand Prix des Sciences mathématiques. Measured in pages, one-sixth of Minkowski's work was written before he was 18. His work of the next ten years deals almost exclusively with quadratic forms.

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* *Bull. Sci. Math*, France, vol. 36 (1912), p. 73.

In the early nineties, when about twenty-eight years old, Minkowski is found opening up his great work on the geometry of numbers, which may be said to occupy about a decade of his life. His book, Geometrie der Zahlen, of which the first part appeared in 1896 in the midst of this decade, was never finished. (The short second part published by Hilbert after the author's death was really merely the conclusion of the fifth section of the first part.) It is to the geometry of numbers with its fundamental concept of convex bodies that we may probably attribute the greatest brilliancy of Minkowski's brilliant career; and to it, also, we may look for his most permanent impressions upon mathematical thought. His work on geometry is a natural corollary of that on the geometry of numbers, and the major part of it in his collected works is the hundred page memoir, previously unpublished, on the theory of convex bodies and in particular on the foundation of the surface-concept as applied to them.

At the close of his life, Minkowski was working on physics. He had printed in the Encyclopedia the article on Capillarity, which is a model of exposition. He had printed his memoir on electromagnetic phenomena in moving bodies, had delivered his address at Cologne on Raum und Zeit, and was preparing a contribution on the deduction of the electromagnetic equations for moving bodies from the point of view of electron theory. This last contribution prepared by M. Born, more from his reminiscences of conversations with the author than from Minkowski's unintelligible notes, is included in the collected works.

Just how much praise the future will attribute to this work on electrodynamics we cannot estimate. The posthumous contribution seems to have had little influence; the preceding memoir is written in a somewhat clumsy notation, appears at times to be translating rather blindly known results into a new notation, and contains some errors.* But the Cologne address is a gem. In it is formulated with satisfactory simplicity the proposition that the laws of physics are fundamentally relations between certain vectors or other geometric elements (affected by coefficients) attached to four-dimensional loci and that the laws which we observe are relations between the projections of those vectors or other elements on our space-time system of reference. This idea was at the basis of the work in the memoir on electromagnetic phenomena, but needed the address to emancipate it.

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* Wilson and Lewis, "Relativity," *Proc. Amer. Acad.*, vol. 48, p. 495 (1912).

There is one point in Minkowski's work which is of negligible importance for that work but which has attracted such comment that it is worth discussing for itself. When introducing his matrical calculations, the author appends a footnote:* "Mann könnte auch daran denken, statt des Cayleyschen Matrizenkalküls den Hamiltonschen Quaternionenkalkül heranzuziehen, doch erscheint mir der letztere für unsere Zwecke als zu eng und schwerfällig." †

Silberstein in published papers and in his book on relativity has shown conclusively that the analytical work on relativity can be carried out with extreme simplicity of notation by the use of quaternions. The most casual comparison of Silberstein's analysis with that of Minkowski will reveal very strikingly the neat quaternion and the clumsy matrix. The possibility of using quaternions lies in the fact that the Lorentz group is a group of rotations (imaginary or non-euclidean) in the four-dimensional manifold of x, y, z, t, and that, as Cayley showed, quaternions may be used to determine four-dimensional rotations by the formulas $q' = QqQ$, $T^2Q = 1$, where q denotes position.

The question, however, remains whether quaternions, though neat, are really appropriate. That they are not is indicated by Silberstein's admission that he had tried a whole year in vain a great variety of quaternion operations for relativistic purposes before discovering Cayley's proposition.‡ But there is a more fundamental reason: A four-dimensional vector is not a quaternion. A four-dimensional vector becomes a quaternion only after the choice of a second (reference) vector which is the scalar axis. Thus in four-dimensional analysis, just as in three dimensions, a quaternion involves two vectors, though in a very different way. The use of quaternions in geometric analysis in four dimensions involves, therefore, an extraneous element, just as the use of cartesian coordinates in geometric analysis in a plane involves elements extraneous to the geometric problem.

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* Vol. 2, p. 375.

† L. Silberstein, Relativity, viii+295 pp., Macmillan, 1914, states on page 150: "Minkowski himself despised Hamilton's calculus of quaternions as 'too narrow and clumsy for the purpose' in question."

‡ *Phil. Mag.*, May, 1912, p. 790.

When using quaternions we have to be very careful to distinguish results which are intrinsically geometric from results which are relative to the direction of reals. This may be illustrated from the theory of complex numbers. The transformation $z' = az + b$, where a, b, z, z' are vectors in a plane (which become complex numbers after the choice of a real axis), is a transformation of similitude, no matter what direction be chosen as the axis of reals, but the transformation is not independent of that choice.

In relativity the time axis is accidental to a particular observer or group of observers and should be chosen *after* the fundamental work is done, not before. The analysis which is really appropriate to the theory of relativity as conceived by Minkowski is Grassmann's. Even a vector analysis (such as that used by Lewis and me, loc. cit.) assumes an origin, which is theoretically "de trop," though practically not much in the way. Is it not unfortunate that Minkowski should have followed the English Cayley, referred to the Scot-Irish Hamilton, and ignored the German Grassmann? Should not some Geheimer Regierungsrat among his colleagues have given him secret directions to avoid such an unpatriotic scientific mésalliance?

E. B. WILSON.

SHORTER NOTICES.

Histoire des Mathématiques. Par CH. BIOCHE. Paris, Belin Frères, 1914. vi+93 pp. Price, 1 fr. 75 c.

IT is one of the strange anomalies in the making of books that France, where the best work in the history of mathematics was done in the eighteenth and early nineteenth centuries, should have done so little in this line in recent years. Montucla, who wrote the first interesting general history of the subject; Delambre and De la Lande who were his worthy successors; Bossut, whose style maintained well the earlier traditions; Libri, writing in France although Italian by birth, and writing with the style of a novelist; Chasles, putting more mathematics into his work than his predecessors,—all these men contributed very notably to the appreciation of the historical development of the science, and set a high standard of style if not always of scholarship. But of late France has produced no general histories of mathematics worthy the name.

To be sure Paul Tannery was a remarkable scholar in his field, and there are few men now living who can rank with Duhem, but the former never attempted a general history and the latter has not done so as yet.

It is for such reasons as these that one looks with special interest to even the simplest efforts to revive the splendid traditions of France in a field that was one time peculiarly her own. And so, while the work of M. Bioche is modest in size and humble in purpose, it is none the less welcome as an evidence of growing interest in the subject. Urged as he says by his " excellents camarades d'Eçole normale, Henri Bergson et Gaston Milhaud," M. Bioche has set about to write a history of ideas rather than one of literature, a record of the stream of progress of mathematics rather than a biography of mathematics.

As a result he has produced a manual somewhat like Mr. Rouse Ball's little Primer of Mathematics in size, although quite different in general treatment. The work consists of eleven chapters, devoted successively to the following topics: Mathematics before the time of the Alexandrian school; the school of Alexandria; the middle ages; the geometry of the Renaissance; the origin of algebra; analytic geometry; the infinitesimal calculus; geometry in the seventeenth and eighteenth centuries; the nineteenth century; ancient astronomy; modern astronomy.

It is not to be expected that new contributions should appear in such a handbook, and there is nothing of this nature to record. A few assertions may be found, however, which are not usually seen in elementary treatises, as that Archytas of Tarentum was probably the first to consider a curve of double curvature, that Aristarchus was the first to make a tentative evaluation of the elements of the solar system, that Apastamba stated the pythagorean theorem before the conquest of Alexander, that Oresmus had the true idea of function in his De latitudinibus formarum, and that the early Greeks used the idea of the climate (the parallel of equal maximum day lengths) instead of the notion of latitude with respect to the equator.

The style of the author maintains the high reputation of the French school and leaves little to be desired. The statements are, however, not altogether free from error; and the omission of a name like that of Mahavir is difficult to explain. The

dates are generally stated as if they were known with certainty, as that Euclid was born in 330 B.C., and Fibonacci in 1175 A.D., while in reality many of these statements are very doubtful and are liable to be put to unfortunate use by the novice. Among the probable errors of statement are the assertion that Heron was a contemporary of Hipparchus, and that Jordanus Nemorarius was the Jordanus who was general of the Dominicans. Among the certain errors are the assertions that Alcuin was abbot of Canterbury, and that Omar Khayyâm was of Arab rather than Persian stock; and among the typographical errors are the printing of Gunther for Günther (page 26), Muller for Müller (page 30), Harriott for the preferred form of Harriot (page 34), and Plucker for Plücker in the index (with a wrong reference). But in spite of these little blemishes the book will serve a good purpose, particularly among the students of the secondary schools of the French-speaking countries.
DAVID EUGENE SMITH.

Solid Geometry. By SOPHIA FOSTER RICHARDSON. Boston, Ginn and Company, 1914. iv + 209 pp.

As the author states in the preface, she gives in this book the "usual course in solid geometry more complete in logical structure than that of the text-books commonly used." Definitions and axioms are quite numerous and prominent and it is by carefully stating these that many difficulties are avoided. For instance there is no difficulty nor incompleteness in the proofs of the theorems about the intersection of a cylinder or cone with a plane through an element and another point of the surface because the theorems are explicitly limited to convex surfaces. We find here also the practice, too rare in American texts, of establishing the existence of a geometric object before defining it. Thus the theorem that a straight line perpendicular to each of two intersecting straight lines at their point of intersection is perpendicular to every straight line in their plane passing through their point of intersection, is given before the definition of a perpendicular to a plane. Similarly the theorem "Any tangent line to a convex cylindrical surface and the element through its point of contact determine a plane which contains no other point of the surface" leads to the definition of a tangent plane to a convex cylindrical surface. As in most texts, geometric locus is defined and the two parts of a locus problem are pointed out, but the notable fact is that the author practices what she preaches, not only in showing that the intersection of two planes is a straight line but also in various other theorems in which too many authors are satisfied with half proofs.

Several theorems on parallel lines and planes are given before theorems on perpendicular lines and planes, an arrangement which would seem to offer less resistance to the beginning student. The exercises throughout the book demand thought, most of them being theorems rather than numerical problems.

The last chapter of the book on "Symmetry and similarity" and the Appendix on "Irrational numbers, variables and limits" offer an elementary treatment of topics to which the student of solid geometry does not usually have access.

RAINARD B. ROBBINS.

A Geometrical Vector Algebra. By T. PROCTOR HALL. Western Specialty, Vancouver, B. C. 30 pp.

HALL'S Geometric Vector Algebra is hardly what its name implies, since trigonometry and determinants are constantly needed to help out the computations of the vector algebra. We fail to attain the promise of the introduction, where we are led to expect a calculus of the space elements themselves. The exposition of principles in the first nine pages is fairly sound, although fragmentary. The word "tensor" is not defined, but means "length of a vector." S_{ab} and V_{ab} mean, substantially, the dot and cross products of Gibbs. But the product of two vectors is defined as a new vector, which turns out to consist of the usual vector product plus another vector term having the direction of the multiplying vector and the length S_{ab}. Evidently with this multiplication not much progress can be made toward a working system. The author himself proves that multiplication is not commutative, associative, nor, in general, distributive. The square of a unit vector, by the definition, is equal to the vector itself, but as a perpendicular operator the square is negative unity; whence equals cannot be put for equals.

Trouble comes in the attempt to show that quaternions are special cases of these operations. The first three fundamental characters of a quaternion on page 10 are correctly stated, but the fourth does not hold, for it contradicts the proof on page 6 that an operator is not distributive. We may indeed resolve the vector into components, but to say that, as an operator, the vector can be expressed as the sum of three parts, but cannot be distributed with respect to these parts, has no meaning. A non-distributive operator is in no sense a quaternion, since it does not obey the laws of quaternions (which are associative and distributive), and since no operational meaning can here be attached to the quadrinomial form so important in quaternions.

The reader who studies out the problems of the remaining 20 pages will do well to avoid an increasing sense of irritation. Very little use is made of the principles first laid down; but a great deal of a mysterious looking but trivial notation for loci, in no way a necessary part of the system. The problem work has no advantage in compactness of reasoning over the usual analytic geometry. In this respect it endures no comparison with the pages of Hamilton or Grassmann, of Heaviside or Gibbs. No doubt it is too much to expect such a test of a brief monograph, but one naturally assumes the problems to have been chosen so as to show the method at its best.

FRANK L. HITCHCOCK.

Mechanics of Particles and Rigid Bodies. By J. PRESCOTT. London, Longmans, Green, and Company, 1913. viii+535 pp.

THIS book has been designed to meet the needs of students aiming for a pass degree at a British university and contains all that they require in the subject of applied mechanics except hydrostatics.

Practically all English texts on mechanics include long lists of problems; many of them consist principally of illustrative examples and problems. This volume, however, presents a systematic development of the theory in which no pains have been spared to make the proofs rigorous enough for pure science, while the practical side of the subject has not been neglected. The problems following each chapter have evidently been chosen with great care and cover a wide range. Some of them demand merely substitution in formulas and numerical calculation, while others offer considerable theoretical difficulty. A special feature is that the answer is given to nearly every question.

An elementary course in the calculus is presupposed for the study of this text and the author appreciates the fact that the student who is applying the calculus for the first time to physical problems has great difficulty in turning his formulas into numbers. Accordingly the examples worked in the text include several numerical ones, and some attention is given to putting results of theory in a form convenient for calculation. The book has a utilitarian bias which should make it useful to the engineering student, as well as interesting and live to the student of pure science. This attitude is emphasized throughout and is especially noticeable in the chapter on motion in two dimensions, where practical numerical applications are made of almost every topic treated, and in the applications to problems of astronomy.

The book begins with the subject of statics and, in addition to usual topics under this head, Part I contains chapters on graphical methods, moments of inertia, potential and attraction, and an extensive chapter on elasticity including strength of materials. Part II is devoted to the dynamics of a particle and the usual special types of motion are treated in excellent style. Besides a fairly complete chapter on central forces there is one on the problem of two or more bodies. The dynamics of a rigid body forms Part III and this is concerned principally with motion parallel to a fixed plane. There is also a chapter on motion in three dimensions of a rigid body with a symmetrical axis. The last chapter is on units and dimensions. If the student is to use the idea of dimensions as a check on his equations, it seems better to place this subject in the earlier part of the book.

W. R. LONGLEY.

Annuaire pour l'An 1915. Publié par le Bureau des Longitudes. Paris, Gauthier-Villars, 1914.

THE Annuaire bears no signs of the struggle which is taking place in Europe. It arrived about the usual time and contains the usual information brought up to date. The astronomical part is increased by a descriptive note, with maps, of the constellations and with the names and positions of the stars which form these groups. The editors frequently find it necessary to add new information and, in order to keep the volume within reasonable limits, to omit less needed tables and articles, publishing the latter only occasionally. This year the geographical positions of the principal towns on the earth are omitted as well as the long table of magnetic elements in France.

The single Notice is a long one by M. G. Bigourdan, "Les méthodes d'examen des miroirs et des objectifs." As is often the case in these appendices, both the older and the more modern methods are described, their mathematical bases are explained and applications to various instruments are made, so that the article practically constitutes a treatise on the subject in a form which can be read and understood without serious difficulty.

E. W. BROWN.

NOTES.

AT the meeting of the London mathematical society on March 11 the following papers were read: By E. W. HOBSON: "Some theorems in the theory of series of orthogonal functions"; by P. A. MACMAHON: "Investigations in the theory of the partition of numbers by a new method of partial fractions"; by R. L. HIPPISLEY: "Reciprocal and parallelogram linkages"; by J. R. WILSON: "A pseudo-sphere the equation of which is expressible in terms of elliptic functions"; by T. C. LEWIS: "Circles, spheres, etc., associated with a triangle, orthocentric tetrahedron, etc."

THE fifth ordinary meeting of the Edinburgh mathematical society for the session was held at the University of Glasgow on March 12. The following papers were read: By A. W. H. THOMPSON: "Solid geometry"; by G. D. C. STOKES: "A simple link apparatus for the mechanical solution of quadratic equations"; by F. TAVANI: "New formulae about the theory of the series of alternate sign."

BEGINNING with the April number, the staff of associate editors of the *Transactions of the American Mathematical Society* will include Professors A. B. COBLE and W. A. HURWITZ in place of Professors J. I. HUTCHINSON and MAX MASON, who have served since 1902 and 1911 respectively.

THE March number of the *Proceedings of the National Academy of Sciences* contains the following mathematical papers: "A note on functions of lines," by G. A. BLISS; "A classification of quadratic vector functions," by F. L. HITCHCOCK. The April number of the *Proceedings* contains:

"Groups possessing at least one set of independent generators composed of as many operators as there are prime factors in the order of the group," by G. A. MILLER; "The second derivatives of the extremal integral for a general class of problems of the calculus of variations," by ARNOLD DRESDEN; "Point sets and allied Cremona groups," by A. B. COBLE; "The straight lines on modular cubic surfaces," by L. E. DICKSON.

THE Theory of Functions of a Complex Variable, by E. J. TOWNSEND, is announced by Henry Holt and Company as in press and to appear in June.

THE Smith prizes of Cambridge University for 1915 have been awarded as follows: to H. GLAUERT, of Trinity College, for his essay "On the elliptical form of a rotating fluid as disturbed by a satellite," and to H. JEFFREYS, of St. John's College, for his essays "Certain hypotheses as to the internal structure of the earth and moon" and "On a possible distribution of meteors." The Rayleigh prize was awarded to J. PROUDMAN, of Trinity College, for his papers on tidal motion.

THE Adams prize commission announces as the subject of the prize competition for 1915–1916 the following:

"The course of evolution of the configurations possible for a rotating and gravitating mass, including the discussion of the stabilities of the various forms."

THE first award of the Ackermann-Teubner memorial prize in mathematics has been made, in accordance with the wish of the founder, to Professor FELIX KLEIN.

THE royal Venetian institute announces the following prize problem for 1917:

To make some notable extension to the theory of periodic solutions of differential systems. It is not always possible to profit by the classical method of Poincaré, that of varying the parameters in a known solution, or at least, only for small variations. When this method does not apply, the only concrete result concerning the condition of existence is the criterion of Whittaker. The most desirable investigation would be the elucidation of the law of distribution of the periodic solutions within the field of the general integral.

The competition is unrestricted. Memoirs should be written in Italian, French, English, or German, and submitted to the institute before December 31, 1917. The value of the prize is 3000 lire.

UNIVERSITY OF STRASSBURG.—The following courses in mathematics are being given during the present summer semester:—By Professor F. SCHUR: Differential geometry, three hours; Selected portions of projective geometry, two hours; Seminar, two hours.—By Professor G. FABER: Theory of functions of a complex variable, three hours; Partial differential equations, two hours; Seminar, two hours.—By Professor M. SIMON: Non-euclidean geometry, two hours.—By Professor J. WELLSTEIN: Differential and integral calculus, three hours; Mechanical integration, two hours.—By Professor L. v. MISES: Descriptive geometry, with exercises, three hours.—By Professor P. EPSTEIN: Analytic geometry of space, three hours; Introduction to determinants, two hours. —By Dr. A. SPEISER: Theory of numbers, two hours.—By Professor J. BAUSCHINGER: New methods in celestial mechanics, three hours.

UNIVERSITY OF CHICAGO.—The following courses are announced for the summer quarter of 1915.—All courses are five hours a week.—By Professor E. H. MOORE: Synthetic projective geometry (first term); General analysis (first term). —By Professor A. C. LUNN: Theory of functions; Theory of relativity.—By Professor G. A. BLISS: Definite integrals; Calculus of variations.—By Professor W. D. MACMILLAN: Differential equations; Celestial mechanics.—By Professor A. DRESDEN: Analytic projective geometry (second term); Determinants and solid analytic geometry.—By Professor J. W. A. YOUNG: Differential calculus; Theory of numbers.—By Professor R. D. CARMICHAEL: Integral calculus; Differential equations (second term).—By Professor E. J. WILCZYNSKI: Projective differential geometry.—By Professor F. R. MOULTON: Theory of rotating bodies.

COLUMBIA UNIVERSITY.—Summer session, July 6 to August 13.—By Professor JAMES MACLAY: Higher algebra; Functions of a complex variable.—By Professor EDWARD KASNER: Theory of geometric constructions; Differential geometry.—By Professor W. B. FITE: Projective geometry; Differential equations.

THE following advanced courses in mathematics are announced for the academic year 1915–1916.

COLUMBIA UNIVERSITY.—By Professor T. S. FISKE: Differential equations, four hours, second half-year.—By Professor F. N. COLE: Theory of groups, three hours, second half-year. —By Professor C. J. KEYSER: Modern theories in geometry, four hours; Mathematics, three hours.—By Professor D. E. SMITH: History of mathematics, four hours.—By Professor JAMES MACLAY: Elliptic functions, four hours, first half-year; Applications of elliptic functions, two hours, second half-year.—By Professor EDWARD KASNER: Differential equations, four hours, first half-year; Seminar in differential geometry, two hours.—By Professor W. B. FITE: Infinite series, three hours, first half-year.—By Professor H. E. HAWKES: Differential geometry of curves, three hours, first half-year.

CORNELL UNIVERSITY.—By Professor J. MCMAHON: Theory of probabilities, three hours.—By Professor J. H. TANNER: Algebraic equations, three hours.—By Professor V. SNYDER: Geometry on an algebraic surface, three hours.—By Professor F. R. SHARPE: Vector analysis, three hours.—By Professor D. C. GILLESPIE: Projective geometry, three hours.—By Dr. C. F. CRAIG: Advanced analysis, three hours.—By Dr. F. W. OWENS: Differential equations, with applications, three hours.—By Dr. J. V. MCKELVEY: Mathematical pedagogy, three hours.—By Professor W. A. HURWITZ: Integral equations, three hours.—By Dr. L. L. SILVERMAN: Coordinate geometry, three hours.—By Dr. J. SLEPIAN: Mechanics, three hours.

HARVARD UNIVERSITY.—All courses meet three times a week throughout the year, except those marked *, which meet for half a year.—By Professor W. F. OSGOOD: Introduction to potential functions and Laplace's equation*; Elastic vibrations, Fourier's series, and Bessel's functions*; Theory of functions of several complex variables.—By Professor M. BÔCHER: Modern geometry and modern algebra; Theory of functions.—By Professor C. L. BOUTON: Differential equations and Lie's theory of continuous groups.—By Professor J. L. COOLIDGE: Subject-matter of elementary mathematics*; Probability*; Projective geometry*; Non-Euclidean geometry*.—By Professor H. N. DAVIS: Second course in dynamics.—By Professor G. D. BIRKHOFF: Infinite series and products*; Problem of three bodies.—By Dr. D. JACKSON: Advanced calculus; Theory of numbers*.—By Dr. G. M. GREEN: Differential geometry of curves and surfaces*; Projective differential geometry*.—By Dr. E. KIRCHER: Properties of polynomials, and invariants*; Algebraic numbers*.—By Dr. G. A. PFEIFFER: Linear partial differential equations of the second order*; Conformal transformations*.

Professor BIRKHOFF and Dr. JACKSON will conduct a fortnightly seminar in analysis.

Courses of research are also offered by Professor OSGOOD in the theory of functions, by Professor BÔCHER in analysis and algebra, by Professor BOUTON in the theory of point transformations, by Professor COOLIDGE in geometry, by Professor BIRKHOFF in differential equations, by Dr. JACKSON in the theory of functions of a real variable, and by Dr. GREEN in differential geometry.

JOHNS HOPKINS UNIVERSITY.—By Professor F. MORLEY: Higher geometry, three hours (first half year); Vector analysis, three hours (second half year); Seminar, one hour.—By Professor A. B. COBLE: Algebraic functions, two hours.—By Professor A. COHEN: Advanced differential equations, two hours; Theory of functions, two hours.

PRINCETON UNIVERSITY.—By Professor H. B. FINE: Algebraic functions, three hours.—By Professor L. P. EISENHART: Mechanics, three hours (first term); Projective geometry, three hours (second term).—By Professor O. VEBLEN: Theory of sets of points, three hours; Coordinate geometry, three hours; Advanced analysis, three hours.—By Professor E. P. ADAMS: Electricity and magnetism, three hours; Analytic mechanics, three hours.—By Professor T. H. GRONWALL: Conformal representation, three hours; Differential equations and advanced calculus, three hours.—By Mr. J. W. ALEXANDER: Birational transformations, three hours (first term).—By Mr. A. A. BENNETT: Elliptic functions, three hours (second term).—By Dr. H. GALAJIKIAN: Differential equations of mathematical physics, three hours (first term).

YALE UNIVERSITY.—By Professor J. PIERPONT: Theory of functions of a complex variable, three hours; Modern analytic geometry, three hours; Modern higher geometry, three hours. —By Professor P. F. SMITH: Continuous groups of transformations, three hours.—By Professor E. W. BROWN: Advanced calculus, three hours; Statics and dynamics, three hours; Advanced dynamics, three hours.—By Professor W. R. LONGLEY: Potential theory and harmonic analysis, three hours; Integral equations, three hours.—By Professor W. A. WILSON: Theory of functions of real variables, three hours.—By Dr. D. D. LEIB: Advanced algebra, three hours.—By Dr. G. M. CONWELL and Dr. H. F. MACNEISH: Differential geometry, three hours.—By Dr. E. J. MILES: Calculus of variations, three hours; Advanced calculus of variations, three hours.

THE *Jahresbericht der Deutschen Mathematiker-Vereinigung* notes the following extraordinary series of anniversaries of German professors of mathematics: M. NOETHER, of Erlangen, and A. WANGERIN, of Halle, celebrated their seventieth birthdays on September 24, and November 18, 1914, respectively. The eightieth birthday of E. SELLING, of Munich, fell on November 5, and the eighty-fifth birthday of M. CANTOR, of Heidelberg, on August 23. H. A. SCHWARZ, of Berlin, and J. THOMAE, of Jena, passed the fiftieth anniversary of their doctorates on August 6, and Professor E. LAMPE, of Berlin, on December 21.

PROFESSOR H. S. WHITE, of Vassar College, has been elected a member of the National academy of sciences.

PROFESSORS J. A. MILLER, of Swarthmore College, and W. F. OSGOOD, of Harvard University, have been elected members of the American philosophical society.

PROFESSOR T. S. FISKE has been appointed administrative head of the department of mathematics at Columbia University, succeeding Professor C. J. KEYSER, who has been relieved of administrative duties at his own request.

AT the University of Oklahoma Professor F. C. KENT has resigned and Dr. H. C. GOSSARD has been appointed instructor in mathematics.

At the University of Michigan Professor L. C. Karpinski has been promoted to a junior professorship of mathematics.

At the Massachusetts institute of technology, Dr. H. B. Phillips has been promoted to an assistant professorship of mathematics.

Mr. C. H. Yeaton has been appointed instructor in mathematics at Dartmouth College.

At Princeton University Dr. H. Galajikian has been appointed instructor in mathematics.—Mr. H. L. Baruch has been appointed assistant in mathematics.

The many friends of Professors Klein and Runge, of Göttingen, will be deeply grieved to learn of the death on the battlefield of the former's youngest son-in-law and the latter's younger son.

Professor F. A. Sherman, who held the chair of mathematics at Dartmouth College from 1871 to 1911, died February 25 at the age of seventy-four years.

Professor A. E. Haynes, who retired from his professorship of mathematics at the University of Minnesota in 1911 after eighteen years service, died on March 12 at the age of sixty-six years.

Book catalogues: W. Heffer and Sons, Cambridge, England, catalogue 132, including about 800 titles in mathematics, physics, and engineering.—Galloway and Porter, Cambridge, England, catalogue 76, short list of mathematical books, 75 titles.—R. W. Lull, 84 Hanover Street, Manchester, N. H., list of 68 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMALDI (U.). Sulla evoluzione della matematica come strumento d'indagine: discorso per l'inaugurazione dell'anno accademico nella r. Università di Modena il 5 novembre, 1914. Modena, Soc. tip. Modenese, 1914. 8vo. 35 pp.

BIELANIN (F. A. P.). See FILIPPOWITSCH (F. W.).

BOLTI (L.). L'infinito. Genova, Formiggini, 1912. 8vo. 5+531 pp.

CARR (H. W.). Henri Bergson: The philosophy of change. London and Edinburgh, Jack; New York, The Dodge Publishing Co., 1912. 92 pp.

CATALOGUE of scientific papers (1884–1900). Compiled by the Royal Society of London. Vol. 14, C-Fittig. Cambridge, University Press, 1915. 4to. 1024 pp. 2£ 10s.

CIANI (E.). Il metodo delle coordinate proiettive omogenee nello studio degli enti algebrici (seguito alle Lezioni di geometria proiettiva ed analitica). Pisa, Spoerri (F. Mariotti), 1915. 8vo. 215 pp.+2 tables. L. 10.00

COFFEY (P.). The science of logic. An inquiry into the principles of accurate thought and scientific method. London, Longmans, 1912. 20+445+7+359 pp.

COUTURAT (R.). Philosophical principles of mathematics. Translated from the French into Russian by B. Koren, under the direction of P. S. Juschkewitsch. St. Petersburg, 1912. 8vo. 260 pp.

FILIPPOWITSCH (F. W.). Guide through mathematical literature. Compiled by F. W. F. with the assistance of F. A. P. Bielanin and S. G. Schiperko. (In Russian.) St. Petersburg, 1912. 8vo. 50 pp.

GUASTELLA (C.). L'infinito. From Annuario della biblioteca filosofica. Palermo, Reber, 1912. 170 pp.

HERBERTZ (R.). Die Philosophie des Raumes. Oeffentl. Hochschul-Vortrag. Stuttgart, W. Spemann, 1912. Gr. 8vo. 38 pp.

HILTON (H.). Homogeneous linear substitutions. Oxford, Clarendon Press, 1915. 8+184 pp. 12s. 6d.

JUSCHKEWITSCH (P. S.). See COUTURAT (R.).

KOREN (B.). See COUTURAT (R.).

LOEWY (A.). Lehrbuch der Algebra. 1ter Teil: Grundlagen der Arithmetik. Leipzig, Veit, 1915. 398 pp. Geh. M. 12.00

SCHIPERKO (S. G.). See FILIPPOWITSCH (F. W.).

SINTZOV (D.). Russian mathematical bibliography. Part 2 for the year 1909. Edited by D. Sintzov. (In Russian.) Odessa, "Mathesis," 1912. 8vo. 15+28+10 pp.

STIELTJES (T. J.). Œuvres complètes de Thomas Jan Stieltjes publiées par les soins de la Société mathématique d'Amsterdam. Tome 1. Groningen, P. Noordhoff, 1914. 4to. 472 pp.

WESTAWAY (F. W.). Scientific method: its philosophy and its practice. London, Blackie, 1912. 21+439 pp.

WILDBRETT (A.). Algebraische Analysis, Algebra und Infinitesimalrechnung. 2ter Teil: Infinitesimalrechnung. Nürnberg, Korn, 1914. 199 pp. Geb. M. 3.80

ZAREMBA (S.). Arytmetyka teoretyczna. Krakau, Verlag der K. K. Akademie der Wissenschaften, 1912. 8vo. 20+859 pp.

II. ELEMENTARY MATHEMATICS.

BAKER (W. M.) and BOURNE (A. A.). A shilling arithmetic. London, Bell, 1915. 8vo. 1s.

BARRACLOUGH (A.). See CRACKNELL (A. G.).

BÖRNER (—.). See SUPPANTSCHITSCH (R.).

BOURNE (A. A.). See BAKER (W. M.).

CRACKNELL (A. G.) and BARRACLOUGH (A.). Junior algebra. London, University Tutorial Press, 1915. 6+280 pp. 2s. 6d.

GOTHE (G.). Rechenbuch fur Stadtschulen. Leipzig, G. Freytag, 1912. 8vo. Unterstufe, 1. Heft, 46 pp. Ausgabe für Knabenschulen, I. Teil, 2. Heft, 127 pp. Ausgabe für Knabenvolksschulen, II. Teil, Mittelstufe, 216 pp. III. Teil, Oberstufe, 184 pp. Ausgabe für Knabenmittelschulen, II. Teil, 242 pp. III. Teil, Oberstufe, 206 pp. Ausgabe für Mädchenschulen, I. Teil, Unterstufe, 2. Heft, 127 pp.; II. Teil, Mittelstufe, 215 pp. Ausgabe fur Madchenvolksschulen, III. Teil, Oberstufe, 192 pp. Ausgabe fur Madchenmittelschulen, III. Teil, Oberstufe, 214 pp.

HOFFMANN (T.). Ueber fehlerhafte mathematische Ausdrucksweise. Progr. 798, Realgymn. Zwickau, 1912. 29 pp.

IGNATJEV (E. M.). Im Reiche des Scharfsinnes oder Arithmetik für alle. Ein Buch für Haus und Schule. Versuch einer mathematischen Chrestomathie. Band 2. 2te durchges. u. vermehrte Auflage. (Russisch.) St. Petersburg, 1912. 8vo. 290 pp.

JASCHUNSKI (J.). See SIMON (M.).

KNOCHENDÖPPEL (C.). See MÖHLE (F.).

LAISANT (C. A.). Introduction to mathematics. Translated from the French into Russian under the direction of W. W. Mropzek. St. Petersburg, 1912. 16mo. 185+2 pp.

LIETZMANN (W.). See WEINREICH (H.).

MILNE (R. M.). Mathematical papers for admission into the Royal Military Academy and the Royal Military College, September–November, 1914. Edited by R. M. Milne. London, Macmillan, 1915. 8vo. 28 pp. 1s.

MÖHLE (F.) und KNOCHENDÖPPEL (C.). Lehr- und Uebungsbuch der Mathematik fur Lyzeen und hohere Mädchenschulen. 1ter Teil: für Klasse IV und III. Breslau, Hirt, 1915. 176 pp. Geb. M. 2.25

MROPZEK (W. W.). See LAISANT (C. A.).

ROHRBERG (A.). Der mathematische Unterricht in Dänemark. (Berichte und Mitteilungen der Intern. Math. Unterrichts-Kommission, 2te Folge, Nr. 1.) Leipzig, Teubner, 1915. 8vo. 6+54 pp. M. 2.40

SIMON (M.). Didaktik und Methodik des Rechnens und der Mathematik. Uebersetzt von J. Jaschunski. (Russisch.) St. Petersburg, "Physice," 1912. 8vo. 4+257 pp.

SSAMOCHWALOV (P.). Zum Unterricht der Analysis in Kadettenkorps. St. Petersburg, 1912. 8vo. 28 pp.+8 tables.

SUPPANTSCHITSCH (R.) und BÖRNER (—.). Mathematisches Unterrichtswerk, für Mädchenschulen bearbeitet. Wien, Tempsky, 1914. Arithmetik für Mädchenlyzeen, Heft 4, 106 pp. Heft 5, 63 pp. Geb. Kr. 1.80 + 1.10.

WEINREICH (H.). Die Fortschritte der mathematischen Unterrichtsreform seit 1910, sowie Lietzmann (W.), Der Pariser Kongress der International Math. Unterrichts-Kommission vom 1.–4. April, 1914. Leipzig, Teubner, 1915. Pp. 219–310. M. 3.00

WOLFF (G.). Der mathematische Unterricht in England. (Berichte und Mitteilungen der Intern. Math. Unterrichts-Kommission, 2te Folge, Nr. 2). Leipzig, Teubner, 1915. 8vo. 6 + 205 pp.

III. APPLIED MATHEMATICS.

AUERBACH (F.). See WARBURG (E.).

BARTLETT (D. P.). General principles of the method of least squares, with applications. 3d edition. Boston, D. P. Bartlett, 1915. 8vo. 142+12 pp. Cloth. $2.25

BRAUN (F.). See WARBURG (E.).

COAN (C. A.). See COLMAN (S.).

COLMAN (S.). Nature's harmonic unity; a treatise on its relation to proportional form, with mathematical analysis by C. A. Coan. New York, Putnam, 1912. 16+327 pp. 8vo.

DREWS (A.). See MOROSOW (N.).

EINSTEIN (A.). See WARBURG (E.).

FANO (G.). Lezioni di geometria descrittiva, date nel r. politecnico di Torino. 2a edizione. Torino, Pavia, 1914. 8vo. 19+455 pp. L. 16.00

HERGLOTZ (G.). Ueber die analytische Fortsetzung des Potentials ins Innere der anziehenden Massen. (Preisschriften gekrönt und herausgegeben von der Fürstlich Jablonowskischen Gesellschaft zu Leipzig, math.-naturw. Sektion, Nr. 19.) Leipzig, Teubner, 1914. 8vo. 52pp. M. 3.60

HOFFMANN (B.). Mathematische Himmelskunde und niedere Geodäsie an den hoheren Schulen. (Abhandlungen der Intern. Math. Unterrichts-Kommission, III, 4.) Leipzig, Teubner, 1912. 6 + 68 pp.

JADANZA (N.). Trattado di geometria pratica: lezioni date nel r. politecnico di Torino. 2a edizione, corretta ed ampliata. Torino, Bona, 1914. 8vo. 12+808 pp.

LANGE (M.). Das Schachspiel und seine strategischen Prinzipien. (Aus Natur und Geisteswelt, Band 281.) 2te Auflage. Leipzig, Teubner, 1914. 12mo. 108 pp. Geb. M. 1.25

LOHMANN (H.). Die technischen Hülfsmittel für den physikalischen und mathematischen Unterricht am Konig Georg-Gymnasium nebst einem Bericht über Orts- und Hohenbestimmungen im Schulgrundstucke. (Progr. 766.) Dresden, 1912. 48 pp.

LORENTZ (H. A.). See WARBURG (E.).

MOROSOW (N.). Die Offenbarung Johannis. Eine astronomisch-historische Untersuchung. Mit einem Geleitwort von A. Drews. Stuttgart, W. Spemann, 1912. Lex. 8vo. 229 pp.+1 table.

PAPERS set in the qualifying examination for the Mechanical Sciences Tripos, 1906–1913. Cambridge, University Press, 1915 90 pp. 2s.

RICHTER (G.) Bewegung, die vierte Dimension. Philosophische Grundlagen der Wissenschaft. Ein Versuch. Wien, W. Braumüller, 1912. 8vo. 12+140 pp.

RUSSISCHE Bibliographie der Naturwissenschaften und Mathematik. Von der Akademie der Wissenschaften herausgegeben. Band V (1906). St. Petersburg, 1912. 8vo. 6+126 pp.

TABLES annuelles de constantes et données numériques de chimie, de physique et de technologie. Vol. III, année 1912. Paris, Gauthier-Villars, 1914. 52+595 pp. Fr. 35.00

TOWNSEND (J. S.). Electricity in gases. Oxford, Clarendon Press, 1914. 8vo. 16+496 pp. 14s.

WARBURG (E.). Physik, unter Redaktion von E. Warburg. Bearbeitet von F. Auerbach, F. Braun, A. Einstein, H. A. Lorentz, etc. (Kultur der Gegenwart, III. Teil, III. Abt., 1. Band.) Leipzig, Teubner, 1915. Lex. 8vo. 8 + 762 pp. Calf. M. 26.00

YOUNG (A. E.). Practical mathematics. Second year. London, Routledge, 1915. 11+164 pp. 8vo. 2s.

L RARY.

THE APRIL MEETING OF THE SOCIETY AT CHICAGO.

THE thirty-fifth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 2–3, 1915, it being the fourth regular meeting of the Society at Chicago. Seventy-five persons were in attendance upon the various sessions, including the following fifty-three members of the Society:

Professor G. A. Bliss, Professor P. P. Boyd, Professor J. W. Bradshaw, Dr. Josephine E. Burns, Professor R. D. Carmichael, Professor A. F. Carpenter, Dr. E. W. Chittenden, Dr. G. R. Clements, Professor H. E. Cobb, Mr. L. C. Cox, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor L. E. Dickson, Mr. C. R. Dines, Professor Arnold Dresden, Professor W. B. Ford, Professor A. B. Frizell, Professor E. R. Hedrick, Professor O. D. Kellogg, Dr. A. J. Kempner, Professor Kurt Laves, Professor C. E. Love, Dr. W. V. Lovitt, Professor A. C. Lunn, Dr. E. B. Lytle, Professor Malcolm McNeill, Professor W. D. MacMillan, Dr. T. E. Mason, Professor G. A. Miller, Professor C. N. Moore, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Professor S. W. Reaves, Professor H. L. Rietz, Mr. George Rutledge, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Professor C. H. Sisam, Professor H. E. Slaught, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleck, Dr. G. E. Wahlin, Professor Mary E. Wells, Professor W. D. A. Westfall, Professor E. J. Wilczynski, Professor D. T. Wilson, Professor A. E. Young, Professor J. W. A. Young, Mr. C. H. Yeaton, Professor Alexander Ziwet.

Professor F. R. Moulton, Vice-President of the Society, presided at the sessions on Friday morning and Saturday afternoon, and Professor E. J. Wilczynski at those on Friday afternoon and Saturday morning.

The social gathering and dinner on Friday evening was attended by forty-seven members.

The following papers were presented at this meeting:

(1) Professor C. H. SISAM: "On sextic surfaces having a nodal curve of order nine."

(2) Professor A. F. Carpenter: "A theorem on the flecnode curves of a ruled surface."

(3) Professor C. T. Sullivan: "Analytic characterization of surfaces having a degenerate directrix quadric."

(4) Professor L. E. Dickson: "Modular quartic curves."

(5) Dr. W. V. Lovitt: "A type of singular points for a transformation on three variables."

(6) Dr. W. V. Lovitt: "Singularities of contact transformations in two-space."

(7) Professor E. J. Wilczynski: "Relations between the theory of congruences and the theory of surfaces."

(8) Address by former chairman of the Chicago Section Professor E. H. Moore: "Report on integral equations in general analysis."

(9) Dr. G. E. Wahlin: "A new development of the theory of algebraic domains."

(10) Professor G. A. Bliss: "Jacobi's condition for problems of the calculus of variations."

(11) Professor C. E. Love: "On linear difference and differential equations."

(12) Mr. W. L. Hart: "Theorems on functions of infinitely many variables" (preliminary report).

(13) Professor F. R. Moulton: "The solution of an infinite system of differential equations" (preliminary report).

(14) Professor C. N. Moore: "On the summability of certain types of series."

(15) Professor R. D. Carmichael: "On the solutions of linear homogeneous difference equations."

(16) Professor R. D. Carmichael: "On certain problems in diophantine analysis."

(17) Professor E. B. Stouffer: "On seminvariants of linear homogeneous differential equations."

(18) Professor W. B. Ford: "On the representation of arbitrary functions by definite integrals."

(19) Professor W. D. MacMillan: "The convergence of the power series $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{a_{ij}}{i-j\gamma}x_1^i x_2^j$."

(20) Dr. E. W. Chittenden: "On equivalence of K relations."

(21) Dr. E. W. Chittenden: "On the existence of continuous functions."

(22) Professor L. E. Dickson: "Historical note on the

proof of the quadratic reciprocity law in a posthumous paper by Gauss."

(23) Professor G. A. MILLER: "Independent generators of a group of finite order."

(24) Professor A. B. FRIZELL: "Well ordered sets of infinite permutations."

(25) Mr. C. R. DINES: "On the Hilbert-Schmidt theory in general analysis and the generalization of a theorem by Picard."

(26) Mr. C. R. DINES: "On the instance suggested by the analogy of the sphere and the ellipsoid as regards positive kernels."

(27) Professor E. J. MOULTON: "On the deviation of a rotating compressible fluid mass from a true spheroid."

(28) Professor R. D. CARMICHAEL: "On the use of functional equations in diophantine analysis."

The papers of Professors Sullivan, Stouffer, and Miller, and the first paper by Dr. Lovitt were read by title. Mr. Hart was introduced by Professors Moore and Moulton. Abstracts of the papers follow, the numbers corresponding to those in the list of titles above.

1. In this paper, Professor Sisam points out some fundamental properties of the unruled sextic surface having a nodal curve of order nine, and of the sextic surface in space of six dimensions of which it is the projection.

2. By determining the conditions satisfied by the coordinates of a plane fixed in space, when referred to a moving tetrahedron of reference, Professor Carpenter proves the following theorem: Let an arbitrary fixed plane P cut the generators of a ruled surface S and let π be the pole of P with respect to the quadric osculating S along a generator g. Then the tangent at π to the locus of π cuts g in a point which is harmonically separated from the point in which P cuts g by the flecnodes of g. A special case of this theorem is stated by M. A. Demoulin in the *Comptes Rendus*, June, 1908, page 1381.

3. The integral surfaces of a completely integrable system of partial differential equations

$$\frac{\partial^2 y}{\partial u^2} + 2b\frac{\partial y}{\partial v} + fy = 0, \quad \frac{\partial^2 y}{\partial v^2} + 2a'\frac{\partial y}{\partial u} + gy = 0,$$

where the fundamental invariants a' and b are characterized by the value

$$a' = b = \frac{1}{2} \frac{\sqrt{\left(\frac{dU}{dy}\right)\left(\frac{dV}{dv}\right)}}{(U + V)},$$

constitute the class of surfaces whose asymptotic curves belong to linear complexes. In a former paper (in volume 15 of the *Transactions*), Dr. Sullivan studied these surfaces and showed that they were the envelopes of either of two one-parameter families of ruled surfaces $[R(u), R(v)]$ with straight line directrices; and that the loci of the directrices of $[R(u), R(v)]$ were complementary reguli of a quadric surface (directrix quadric).

The present paper is devoted to the exceptional case where the directrix quadric degenerates into two planes, distinct or coincident. It is shown that the nature of the directrix quadric is completely characterized by the forms of the two invariants θ/a'^2 and θ'/b^2, which played such a fundamental rôle in the determination of the canonical forms of the seminvariants $f(u, v)$ and $g(u, v)$.

It is found that the most general forms of these invariants are:

1. Directrix quadric non-degenerate,

$$\left(\frac{\theta}{a'^2}\right) = \frac{a_1U^2 + b_1U + c}{(dU/du)}, \quad \left(\frac{\theta'}{b^2}\right) = \frac{a_1V^2 - b_1V + c}{(dV/dv)},$$

where a_1, b_1 and c are constants.

2. Directrix quadric degenerates into two distinct planes,

$$a_1 = b_1 = 0, \quad c \neq 0.$$

3. Directrix quadric degenerates into two coincident planes,

$$a_1 = b_1 = c = 0.$$

The plane of case 3 is identical with that determined by the plane net of lines of the degenerate directrix congruence of the first kind discussed in section 7 of the paper cited above. In this case the finite equations of the surfaces have been obtained by Professor Wilczynski (“Flächen mit unbestimmten Direktrixkurven,” *Mathematische Annalen*, December, 1914).

4. Professor Dickson's paper will appear in the current volume of the *American Journal of Mathematics*.

5. It is the purpose of Dr. Lovitt's first paper to discuss from a geometrical standpoint, as far as possible, the character of a transformation

$$(1) \qquad x = \phi(u, v, w), \quad y = \psi(u, v, w), \quad z = \chi(u, v, w)$$

in the neighborhood of a special type of point at which the Jacobian of the transformation vanishes.

The initial assumptions are such that attention is confined to non-singular points both of the surface defined by the Jacobian of the transformation and of the transform of this surface by means of equations (1). In section two it is shown that under our assumptions there is a one-to-one correspondence between either of two regions in *uvw*-space on opposite sides of the Jacobian surface J and their transforms in the *xyz*-space, the transformed regions being on the same side of a surface J_1 which is the transform of the surface J by means of equations (1). Moreover each of the transformed regions is a connected region possessing interior points. The equations of transformation define inverse functions which are continuous at points of the *xyz*-space which are on the surface J_1. In section three most of the results of the preceding section are extended to a transformation of a neighborhood of a portion of a singular surface not necessarily restricted to a neighborhood of a particular point. In section four most of the results of the preceding section are extended to a transformation in n variables.

6. In his second paper, Dr. Lovitt studies the effect of the transformation

$$x_1 = X(x, y, p), \quad y_1 = Y(x, y, p), \quad p_1 = P(x, y, p)$$

upon the contact of curves

$$(1) \qquad x = f(t), \quad y = g(t), \quad y' = g'/f' = h(t)$$

in the neighborhood of a set of values $(x, y, p = x_0, y_0, p_0)$ for which the functional determinant of the transformation vanishes, but the matrix of the determinant is of rank two. Let the transformed curves be given by

$$x_1 = f_1(t), \quad y_1 = g_1(t), \quad y_1' = g_1'/f_1' = h_1(t),$$

where f_1, g_1, h_1 are defined by means of the equations of transformation and equations (1). Let us designate as critical curves that family of curves which yield the values of $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ necessary to make

$$f_1^{(j)} = h_1^{(j)} = 0 \quad \text{for} \quad j = 1, 2, \cdots, k-1,$$

while $[f_1^{(j)}]^2 + [h_1^{(j)}]^2 \neq 0$.

Then the necessary and sufficient condition that two of the critical curves shall be transformed into curves with second order contact is that the critical curves have contact of order $k + 1$.

7. In a paper presented to the Society at Chicago in December, 1914, Professor Wilczynski introduced a number of new concepts into the theory of congruences, among which the axis and ray congruences, and the axis and ray curves were especially important. On the other hand he has introduced into the theory of surfaces the notion of the directrix congruences and the directrix curves. In the present paper the theory of congruences is enriched by formulas and theorems expressing the relations which exist between the directrix curves and directrix congruences of the focal surface, and the other properties of the congruence.

9. In volume 144 of *Crelle's Journal* Hensel published a paper entitled "Ueber die Grundlagen einer neuen Theorie der quadratischen Zahlkörper." Dr. Wahlin's paper is a generalization of this development to any algebraic domain of degree n.

In the first place the ring $R(q, \alpha)$, consisting of the g-adic algebraic numbers determined by the root α of an equation of degree n, is considered. It is shown how the study of these numbers can be reduced to the case where g is a rational prime p and we have the ring $R(p, \alpha)$.

A method is then devised by means of which the further study of this ring can be reduced to the study of certain domains constructed out of the numbers of $R(p, \alpha)$ with a different condition for equality.

10. There are two well-known methods of deducing Jacobi's necessary condition in the calculus of variations. One is a geometric proof not easily extensible to higher spaces, which excludes important exceptional cases; the other depends upon complicated manipulations of the second variation. For the problem in parametric form in the plane the reduction

of the second variation was devised by Weierstrass and is a remarkable piece of analysis. It is, however, very artificial, and also unnecessary if one has in view only the proof of Jacobi's condition. For problems in parametric form in higher spaces a discussion of the second variation has been made by von Escherich by methods in part quite unsymmetrical. The lack of symmetry is due to the division of an arc

$$y_i = y_i(t) \quad (t_1 \leqq t \leqq t_2;\ i = 1, 2, \cdots, n)$$

into a finite number of pieces on each of which at least one of the derivatives $y_i'(t)$ is different from zero, a device which leads to inelegant complications, though his results are symmetric in form. In the paper of Professor Bliss a proof of Jacobi's condition is given which applies with equal simplicity to the parametric case in the plane or in higher spaces. It makes use of the second variation without reduction of any sort, is symmetric in all the variables, and includes the exceptional cases which probably would with difficulty be covered by an extension to higher spaces of the geometric proof mentioned above.

11. The first part of Professor Love's paper is an extension of certain theorems of Ford* on linear difference equations. The behavior, for large positive integral values of x, of n linearly independent solutions of the equation

$$Y(x) = [a_0 + \alpha_0(x)]y(x + n) + [a_1 + \alpha_1(x)]y(x + n - 1) + \cdots + [a_n + \alpha_n(x)]y(x) = 0$$

is determined, provided the functions $\alpha_0(x), \alpha_1(x), \cdots, \alpha_n(x)$ vanish at infinity to a sufficiently high order. It should be noted that no restriction whatever is laid upon the roots of the characteristic equation

$$a_0\mu^n + a_1\mu^{n-1} + \cdots + a_n = 0,$$

except that a_0 and a_n are assumed different from zero.

The second part of the paper contains the parallel investigation for linear differential equations. The same problem under narrower conditions has been treated by Dini.†

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* *Annali di Matematica*, ser. 3, vol. 13 (1907), pp. 263–328.

† *Annali di Matematica*, ser. 3, vol. 2 (1899), pp. 297–324; *ibid.*, vol. 3 (1900), pp. 125–183.

In each part of the paper the results obtained are also applied to the study of the non-homogeneous equation

$$Y(x) = X(x),$$

under suitable restrictions on the function $X(x)$.

12. By a function of infinitely many real variables will be understood a function $f(\xi)$ where ξ represents $(x_1, x_2, \cdots)$. In much of the current work, the function f is defined for values ξ for which $\Sigma_{i=1}^{i=\infty} | x_i^2 |$ converges. In Mr. Hart's paper the functions considered are supposed to be defined for values ξ satisfying

$$M_1^{(i)} \leqq x_i \leqq M_2^{(i)} \quad (M_1^{(i)}, M_2^{(i)} \leqq M; i = 1, 2, \cdots).$$

The function f is said to be completely continuous at the point ξ_0 if, whenever

$$\lim_{n=\infty} x_{i,n} = x_{i,0} \qquad (i = 1, 2, \cdots),$$

then

$$\lim_{n=\infty} f(\xi_n) = f(\xi_0) \quad (\xi_n = x_{1,n}, x_{2,n}, \cdots).$$

In addition to certain general theorems on such continuous functions, among which is included an extension of the mean value theorem, there is considered the solution, for ξ in terms of u, of the infinite system of equations

$$(1) \qquad f_i(\eta) = 0 \quad (\eta = u, x_1, x_2, \cdots; i = 1, 2, \cdots),$$

where the values of η lie in the region R defined by

$$| \eta - \eta_0 | \leqq d_0.$$

The fundamental theorem of implicit function theory is obtained for (1) by a method of successive approximations.

13. Professor Moulton treats an infinite system of differential equations of the analytic type, and, under suitable hypotheses, establishes the existence of an analytic solution. The solution is also studied as a function of the parameters which may be involved and of the initial conditions. The paper is incomplete in that the details of the extension of the solution to the boundary of the region where the initial equations are defined and regular have not been fully worked out.

14. Chapman has obtained sufficient conditions that a series summable (Ck) for any $k > 0$ should be reduced to convergence when certain convergence factors are introduced into the terms of the series. Somewhat earlier G. H. Hardy discussed the conditions under which a series summable (Ck), where k is a positive integer, should remain summable (Ck) when the terms are multiplied by factors of a certain type. In Professor Moore's paper sufficient conditions are obtained that a series summable (Ck) for any $k > 0$ should remain summable (Ck) when certain factors are introduced into the terms.

15. In his first paper Professor Carmichael gives a new method of deriving the general existence theorems for the case of the linear homogeneous difference equation

$$f(x+n) + a_1(x)f(x+n-1) + \cdots + a_n(x)f(x) = 0$$

in which the coefficients $a_1(x), \cdots, a_n(x)$ have the character of rational functions at infinity. By means of the descending formal power series solutions which this equation is known to possess in general a separation of it into two members is effected so that the method of successive approximation is available for deriving a single solution. By repeated use of a method of transformation the remaining solutions of two fundamental sets are then obtained. The analytic character of the functions in one of these sets and their asymptotic character at infinity are determined.

Of the previously developed methods for dealing with the problem of this paper that of the author's dissertation is most closely related to the present one, but the two methods are essentially distinct.

16. By means of a class of algebraic relations due to Fermat and Lagrange and developed in part by Legendre, Professor Carmichael, in his second paper, suggests a general method by which large classes of problems in diophantine analysis become amenable to systematic treatment. The method is applied to the resolution of a considerable number of diophantine equations of interest. The essential element of the method may be described thus: The problem of solving a diophantine equation consists essentially in determining numbers of a certain class (those defined by the form of one member) so that they shall have an additional property (that determined

by the form of the other member). The method suggested and employed in the present paper is suitably to extend the set of numbers defined by the form in one member (sometimes by those in both) so that the extended set shall have the property that the product of any two numbers in the set shall also be in the set. This gives rise to a generalized problem. The latter is solved and the result is then specialized so as to afford the solution of the original problem. This method, though very simple in its characteristics, leads to a considerable number of results of interest.

17. Consider the system of linear homogeneous differential equations

$$y_i'' + \sum_{k=1}^{n}(2p_{ik}y_k' + q_{ik}y_k) = 0, \quad (i = 1, 2, \cdots n),$$

where p_{ik} and q_{ik} are functions of the independent variable x. The most general transformations leaving such a system unchanged in form are given by

$$(1) \qquad y_i = \sum_{k=1}^{n}\alpha_{ik}\bar{y}_k, \qquad |\alpha_{ik}| \neq 0, \quad (i = 1, 2, \cdots n),$$

$$(2) \qquad x = \xi(x),$$

where α_{ik} and ξ are arbitrary functions of x. A function of p_{ik} and q_{ik} and their derivatives which has the same value for the given system as for any system obtained from it by the transformation (1) is called a seminvariant. Professor Stouffer obtains for each value of n a single seminvariant from which the complete system for that value of n may be obtained by successive applications of certain operators. These operators are similar to the Aronhold operator for algebraic invariants. This method of obtaining the seminvariants avoids the solution of complicated systems of partial differential equations.

18. Let $f(x)$ be an arbitrary function of the real variable x defined throughout the interval $(-\pi, \pi)$. Having assigned the special value α $(-\pi < \alpha < \pi)$ to x, let the Fourier series for $f(\alpha)$ be formed. The sum of its first $n + 1$ terms may be put in the form

$$(1) \qquad S(n, \alpha) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\psi(n, x - \alpha)dx,$$

where

$$\psi(n, x - \alpha) = \frac{\sin \frac{1}{2}(2n + 1)(x - \alpha)}{2 \sin \frac{1}{2}(x - \alpha)},$$

and it is a familiar fact that we then have, at least in general,

$$\lim_{n=\infty} S(n, \alpha) = \frac{f(\alpha - 0) + f(\alpha + 0)}{2}. \tag{2}$$

The following general problem is thus suggested: Given an arbitrary function $f(x)$ defined throughout any finite interval (a, b) and consider the integral

$$I(n, \alpha) = \int_a^b f(x)\varphi(n, x - \alpha)dx, \tag{3}$$

where $\varphi(n, x - \alpha)$ is now regarded as an unknown function to be so determined that we shall have, analogously to (2),

$$\lim_{n=\infty} I(n, \alpha) = \frac{f(\alpha - 0) + f(\alpha + 0)}{2} \quad (a < \alpha < b). \tag{4}$$

What conditions upon $\varphi(n, x - \alpha)$ will suffice to insure (4)? This problem was first considered and answered to some extent by Du Bois-Reymond, later it was treated in great detail by Dini in his "Serie di Fourier," and of late years it has been the object of a paper by Hobson.* The conditions in question have thus been in large measure determined, but it does not appear that explicit forms for $\varphi(x, n - \alpha)$ that satisfy such conditions have been worked out to any appreciable extent. In this connection, Professor Ford's paper shows how an infinity of such functions $\varphi(n, x - \alpha)$ may be explicitly obtained.

19. In this paper by Professor MacMillan it is shown that if the series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} x_1^i x_2^j$$

is convergent for $| x_1 | < x_{10}$, $| x_2 | < x_{20}$, then the series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{a_{ij}}{i - j\gamma} x_1^i x_2^j$$

also is convergent if $| x_1 | < x_{10}$, $| x_2 | < x_{20}$, and γ is a positive irrational number which satisfies certain mild conditions.

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* *Proc. London Math. Soc.*, vol. 6 (1908), pp. 349–374.

Two applications of these series are given. The first application is to the function

$$W = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\lambda^i \mu^j}{i - j\gamma},$$

for which the positive real axis is a line of essential singularities. It is shown that this function is continuous across this axis at such irrational points. The second application is to the existence of a solution of the linear partial differential equation

$$x_1 \frac{\partial \varphi}{\partial x_1} - \gamma x_2 \frac{\partial \varphi}{\partial x_2} = p_1 \varphi + p_2,$$

where p_1 and p_2 are convergent ordinary power series in x_1 and x_2. It is shown that solutions exist as ordinary power series in x_1 and x_2.

20. For a class $\mathfrak{P}$ relations Kp_1p_2m, $\overline{K}p_1p_2m$ are supposed to be defined. These relations are of the type K_2 of Moore's Introduction to General Analysis. Dr. Chittenden defines equivalence for the relations K, $\overline{K}$ and shows that for any unsymmetric K possessing the properties 167 as defined by T. H. Hildebrandt* a symmetric equivalent relation $\overline{K}$ may be defined. The theory of classes $\mathfrak{P}$ and functions on classes $\mathfrak{P}$ as developed by Hildebrandt for an unsymmetric K^{167} relation is equivalent to the corresponding theory for symmetric K relations.

21. Hans Hahn has shown that there exists a non-constant continuous function on every class $\mathfrak{P}$ which admits a definition of voisinage.† Dr. Chittenden shows by modifying the methods of Hahn that this result may be extended to classes which admit a definition of distance subject to the condition $L_n(p_n, q_n) = 0$, $L_n(p_n, p) = 0$ imply $L_n(q_n, p) = 0$. It follows, as shown by Hahn, that, if limit is unique, a necessary and sufficient condition that every continuous function on $\mathfrak{P}$ be bounded and attain its bounds is that $\mathfrak{P}$ be extremal. This result is extended to the case of non-unique limits.

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* *American Journal of Mathematics*, vol. 34 (1912), pp. 243–4.

† *Monatshefte fur Mathematik und Physik*, vol. 19 (1908), p. 251–5.

22. The seventh proof by Gauss of the quadratic reciprocity theorem occurs in his posthumous paper, Werke 2, 1863, pages 233–5 (Gauss-Maser, pages 623–4). Gauss left only a meager outline of a possible proof, not revised by him for publication. In the version by Bachmann, Niedere Zahlentheorie, 1, 1902, pages 396–9, resort is had (page 399) a second time to the cyclotomic theory, whereas one needs only the fact that in all cases $S(i)$ and $S_1(i)$ are the roots of the congruence at the bottom of page 398; moreover, Bachmann introduces Galois imaginaries, whereas their use was carefully avoided by Gauss (Disquisitiones Arithmeticae, article 338; Werke, 2, page 217). The object of the note by Professor Dickson is to complete the proof in accord with the spirit of the MS. of Gauss. In *Crelle*, 19, 1839, pages 299–306, Schonemann gave a sketch (neither very clear nor full) of a similar proof based on the reality of the roots of the quadratic congruence corresponding to the cyclotomic equation for two periods. Another proof similar to that by Gauss and published three years prior to his is that by V. A. Lebesgue, *Comptes Rendus*, 51, 1860, pages 9–13; it is based on Kummer's theorem on the reality of the roots of the congruence corresponding to a cyclotomic equation for e periods.

23. The term independent generator has been used mainly with reference to abelian groups. Hence it has been commonly used with the restricted meaning that the group generated by any number of a set of these generators has only identity in common with the group generated by the rest of those in the same set. With this restricted meaning Frobenius and Stickelberger proved in 1879 that the number of the independent generators is an invariant of any abelian group whose order is the power of a prime number. It has been proved recently that the number of these independent generators is an invariant of *every* group whose order is a power of a prime when the independent generators are defined as follows: A set of operators $s_1, s_2, \cdots, s_\lambda$ belonging to the group G is called a set of independent generators of G provided that these λ operators generate G but no $\lambda - 1$ of them generate G.

Among the theorems established by Professor Miller in the present paper are the following: A necessary and sufficient condition that a group G contains at least one set of independent generators composed of as many operators as there are prime factors in the order of the group, is that each of the Sylow subgroups of G is abelian and is generated by a set of operators of prime order each of which is transformed only into powers of itself by every Sylow subgroup whose order is a power of a smaller prime number. The number of the groups of order p^m, p being any prime number, which possess a set of $m - 1$ independent generators is either $\frac{3}{2}(m - 1)$ or $\frac{3}{2}(m - 2) + 1$ according as m is odd or even.

24. By a process used in a paper read before the Society at New York, February, 1915, Professor Frizell proves that all well ordered types of order are comprehended in those furnished by the permutations of a simply infinite set. That is, given a well ordered set w, there exists, among the permutations of an ω-series, a set ordinally similar to w.

25. Professor Moore has shown that the general Hilbert-Schmidt theory of the integral equation for the complex-valued hermitian kernel may be secured on the foundation Σ_5; viz.,

$$(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M}^{LCDD_0R \text{ on } \mathfrak{P} \text{ to } \mathfrak{A}};\ \mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*;\ J^{LMHPP_0 \text{ on } \mathfrak{K} \text{ to } \mathfrak{A}}),$$

where $\mathfrak{A}$ is the class of all real or of all complex numbers, $\mathfrak{P}$ is a general class of general elements forming a general range, $\mathfrak{M}$ is a class of single-valued functions on the range $\mathfrak{P}$ to $\mathfrak{A}$ with the properties $LCDD_0R$, $\mathfrak{K}$ is defined as the *-composite of $\mathfrak{M}$ with itself, and J is a general functional operation having properties as indicated. The definite property P_0 of the functional operation is the property which, in the classical instance of integration states that if, for a continuous function ξ, we have $\int_0^1 \xi(s)\bar{\xi}(s)ds = 0$, then $\xi(s)$ is identically zero. Mr. Dines shows that the theory as to the existence of characteristic functions and numbers of a hermitian kernel κ may be obtained on a foundation in which the functional operation is not restricted by postulation of the property P_0. This permits us then to include as an instance that of integration in which the class $\mathfrak{M}$ is taken as the class of all real-valued, bounded functions, integrable with their squares in the sense of Lebesgue. The relation, κ not identically zero, in the classical instance is replaced by the relation $J_{23\ 41}{}^2\kappa\kappa \neq 0$, and the theory is in general the same. The expansion theorem of Hilbert is stated as giving the expansion of $J^2\alpha\kappa\beta$ where α and β are not necessarily in the class $\mathfrak{M}$ but are functions of a class $\mathfrak{N}$ which contains $\mathfrak{M}$ as a subclass, J being operative on $(\mathfrak{N}\mathfrak{N})_L$ to $\mathfrak{A}$. For example, the class $\mathfrak{N}$ may be taken as the class of all functions, whether bounded or not, integrable with their squares in the sense of Lebesgue.

The unsymmetric kernel is then discussed in a generalization in which not only is the functional operation generalized but also the kernel function is no longer on a composite range necessarily formed by one range with itself but on a composite range formed from two conceptually distinct ranges.

Using these results, Mr. Dines secures a generalization of a theorem due to Picard as to conditions for solvability of an integral equation of the first kind and gives an application to the geometry of a function space.

26. As an instance of the basis Σ_5, Professor Moore has cited that suggested by the analogy of the sphere and the ellipsoid. In this instance, given a function ω, a new functional operation $J\omega$ is obtained from that of Σ_5 by defining the functional operation, operating on a function κ, as $J_{13\;42}{}^2\kappa\omega$. Mercer has shown the importance of the positive kernel as to expansion in a uniformly convergent series of characteristic functions. Mr. Dines discusses the positive kernel in the instance cited above. Conditions on a hermitian function ω are obtained sufficient to secure the equivalence for every hermitian function κ of the properties, positive as to J and positive as to J_ω. There follows a discussion of conditions equivalent to these. The latter part of the work is suggested by that of Fischer on the equivalence of the notions, general and closed, as applied to a unitary and orthogonal set of functions from the class of all functions integrable with their squares in the sense of Lebesgue. The foundation for the discussion is Σ_7, defined as follows:

$$(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{N}^{LD_0BC_J \text{ on } \mathfrak{P} \text{ to } \mathfrak{A}};\ \mathfrak{M}^{LCDD_0RB_0\mathfrak{N}};\ \mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*;\ J^{LMHP \text{ on } \binom{(\mathfrak{N}\mathfrak{N})L}{\mathfrak{K}} \text{ to } \mathfrak{A}}).$$

This differs from the foundation Σ_5 in that the functional operation is not restricted by postulation of the property P_0 and that the class $\mathfrak{N}$ is introduced. The property C_J for the class $\mathfrak{N}$ is a closure property, the class being closed as to convergence in the mean.

27. If a homogeneous fluid mass is rotating about an axis with a sufficiently small angular velocity there are two possible oblate spheroid figures of equilibrium, one nearly spherical, Σ_1, and one much flattened at the poles, Σ_2. If the fluid mass is compressible there are two figures of equilibrium nearly oblate spheroids. One approximates Σ_1 and is depressed in middle latitudes below the spheroid having the same polar and equatorial radii. The other approximates Σ_2 and is elevated in middle latitudes. The former theorem was proved by Airy, Callandreau and Darwin; the latter is a new theorem. This is established and a formula for the deviation from a true spheroid is obtained by Professor Moulton. He starts from an assumed relation between the pressure and density in the fluid and by making a compressibility parameter play a fundamental rôle is able to discuss the more flattened figure.

28. In his third paper Professor Carmichael shows how rational solutions of certain functional equations may be employed in solving problems of a certain class in the theory of diophantine analysis. In particular, several problems of Diophantus and Fermat are readily treated. The contents of both this paper and the preceding one by the same author will appear in his forthcoming "Introduction to Diophantine Analysis," to be published by Wiley and Sons.

H. E. Slaught,
Secretary of the Chicago Section.

A GEOMETRIC DERIVATION OF A GENERAL FORMULA FOR THE SOUTHERLY DEVIATION OF FREELY FALLING BODIES.*

BY PROFESSOR WM. H. ROEVER.

(Read before the American Mathematical Society, October 25, 1913.)

Within the last dozen years interest in the problem of the deviations of freely falling bodies seems to have been revived. There is a substantial agreement, among the writers who have treated this subject, as to the magnitude of the easterly deviation, their result being practically the same as that obtained by Gauss more than one hundred years ago. Concerning the southerly deviation, however, there appears to be a diversity of results. This lack of agreement can, for the most part, be accounted for by the different definitions for southerly deviation and the different potential functions for the earth's field of force which have been used in the derivations of the various formulas. In another paper* the author shows that most of these formulas for the southerly deviation are special cases (corresponding to special potential functions and to special definitions) of his general formula

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* See Bulletin, vol. 20, No. 4, p. 175.

$$\text{(I)} \qquad \text{S.D.} = \left\{ 2\omega^2 \sin 2\phi_0 - 5\left(\frac{\partial g}{\partial \xi}\right)_0 \right\} \cdot \frac{h^2}{6g_0},$$

which may be written in the form

$$\text{(II)} \qquad \text{S.D.} = \xi_1 - \xi_2,$$

where

$$\xi_1 = \left\{ 2\omega^2 \sin 2\phi_0 + \left(\frac{\partial g}{\partial \xi}\right)_0 \right\} \cdot \frac{h^2}{6g_0}, \quad \xi_2 = \left(\frac{\partial g}{\partial \xi}\right)_0 \cdot \frac{h^2}{g_0},$$

S.D. representing the quantity which some writers use for southerly deviation and ξ_1 that which others use. Of the other quantities used in formula (I), h denotes the height through which the body falls, ω the angular velocity of the earth's rotation, ϕ_0, g_0, $(\partial g/\partial \xi)_0$ the values at the point P_0 from which the body falls of the astronomical latitude ϕ, the acceleration g due to weight, and the derivative of g with respect to ξ, where ξ represents distance measured to the south at P_0.

Formula (I), besides including as special cases many formulas which are based on particular potential functions, has a practical advantage over any of these formulas. This the following facts will show. In a country which is not very level the experimentally determined values of the derivative $(\partial g/\partial \xi)$ exceed, in magnitude, many times (sometimes ten or more) the values of this derivative which correspond to generally accepted potential functions. Furthermore, $(\partial g/\partial \xi)$ varies in an apparently very capricious manner. It may be positive at one station and negative at a station a few miles distant.† Therefore, formula (I), with the experimentally

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[*] See the author's paper "Recent work on the deviations of falling bodies," soon to be published in *U. S. Monthly Weather Review*, vol. 43.

[†] See *Transactions Amer. Math. Society*, vol. 13, pp. 469–490 (Introduction); *Astronomical Journal*, nos. 670–672, Part II, § 4.

determined local value of the derivative $(\partial g/\partial \xi)$, is capable of accounting more accurately for the effect, on the southerly deviation of a falling body, of a local irregularity in the earth's field of force (such as a mountain) than any formula which is based on a particular (and consequently inadequate) potential function.

The author has already shown that, under the assumption of an asymmetric distribution of the earth's gravitating matter as well as under the assumption of a distribution of revolution, formula (I) is the first and predominating term of a still longer expression for the southerly deviation of freely falling bodies.* In view of the generality and practical nature of formula (I) it seems worth while to derive it by still another method. In this article a purely geometric derivation is given under the assumption of a distribution of revolution.

§ 1. *Definitions.*

Let us assume a set of rectangular axes $O - u, v, z$ (Fig. 1) which are at rest with respect to the solid part of the earth and such that Oz coincides with the earth's axis of rotation, the positive sense of Oz being from the celestial south pole to the celestial north pole. The field of force which determines

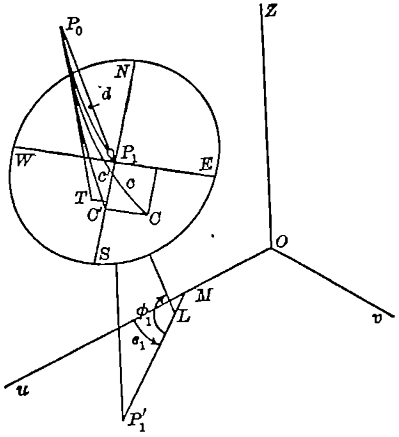

FIG. 1.

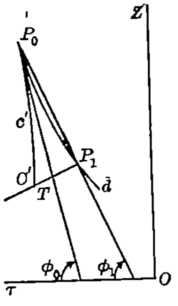

FIG. 2.

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[*] See *Transactions Amer. Math. Society*, vol. 12, pp. 335–353; vol. 13, pp. 469–490; and *Astronomical Journal*, nos. 670–672, Part II.

weight, i. e., the field in which the plumb-line is in equilibrium, is at rest with respect to these axes. We shall call this field of force the *weight field*. Let us denote by P_1 a point (on or near the earth's surface) which is at rest with respect to the axes $O - u, v, z$. The straight line which passes through P_1 and gives the direction of the force of the weight field at P_1 is defined as the *vertical of* P_1. The vertical of P_1 coincides with the string of a plumb-line, the bob of which is situated at P_1.* The *astronomical meridian plane of* P_1 is the plane which passes through the vertical of P_1 and is parallel to the axis of rotation Oz. The *astronomical latitude of* P_1 is the complement ϕ_1 (Fig. 1) of the angle which the vertical of P_1 (to the zenith) makes with the axis Oz (to the celestial north pole). The *astronomical longitude of* P_1 is the angle ϵ_1 (Fig. 1) which the meridian plane of P_1 makes with a fixed plane (zOu say) through Oz, and is measured from 0° to 360° to the east. The *horizontal plane of* P_1 is the plane which passes through P_1 and is perpendicular to the vertical of P_1. The *north-and-south line of* P_1 is the line of intersection of the meridian and horizontal planes of P_1. The *east-and-west line of* P_1 is the straight line which passes through P_1 and is perpendicular to the meridian plane of P_1. This line may also be defined as the intersection of the horizontal plane of P_1 by the plane which passes through P_1 and is perpendicular to the axis Oz.†

Now let us take a point P_0 (Fig. 1) above P_1 and in the vertical P_1P_0 of P_1. The vertical of P_0 will *not* coincide with the vertical of P_1, but it will pierce the horizontal plane $NESW$ of P_1 in a point T which does not coincide with P_1. Let us suppose a material particle to fall, under the influence of the earth's gravitation, from an initial position of rest (with respect to $O - u, v, z$) at P_0. The path of this particle, with respect to the axes $O - u, v, z$, is a curve c which pierces the horizontal plane $NESW$ of P_1 in a point C.

DEFINITION 1. *The perpendicular distance of the point C from the north-and-south line NP_1S of P_1 is called the easterly deviation of the falling particle, and is regarded as positive to the east* (see Fig. 1).

DEFINITION 2. *The perpendicular distance of the point C*

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* It is assumed that the string of the plumb-line is weightless and perfectly flexible, and that the bob is a heavy particle.

† The above definitions are practically the same as those given by Pizzetti, Trattato di Geodesia teoretica, § 5 (1905).

from the east-and-west line EP_1W *of* P_1 *is called the southerly deviation of the falling particle, and is regarded as positive to the south* (see Fig. 1).

The following definitions will also be found convenient. *A line of force of the weight field* is a curve at each point P of which the tangent is the vertical. *A level (or equipotential) surface of the weight field* is a surface at each point P of which the normal is the vertical. The *plumb-bob locus of the point* P_0 is the locus of the bobs of all plumb-lines (of the ideal type defined above) which are supported at P_0. Geometrically this locus may be defined in either of the two following ways:

(1) the locus of the points of tangency of tangents drawn from the fixed point P_0 to the lines of force of the weight field.

(2) the locus of the feet of perpendiculars dropped from the fixed point P_0 to the level surfaces of the weight field.

§ 2. *Some Fundamental Relations.*

Let us denote by ω the angular velocity (with respect to the inertial axes of the solar system) of the earth's rotation, and by V the potential function of the earth due to gravitational attraction alone, i. e., the function defined by the integral

$$V = \kappa \int \frac{dm}{\sqrt{(u-a)^2 + (v-b)^2 + (z-c)^2}},$$

where dm represents that element of mass which is situated at the point whose coördinates, with respect to the axes $O - u, v, z$, are a, b, c, the integration being extended throughout the whole volume of the earth, and κ is the gravitation constant. Then the potential function of the weight field is*

$$W = V + \frac{\omega^2}{2}(u^2 + v^2). \tag{1}$$

Then the equations of the level surfaces of the weight field may be written in the form

$$W = K,$$

where K has constant values. The differential equations of the lines of force of the weight field are

$$\frac{du}{W_u} = \frac{dv}{W_v} = \frac{dz}{W_z}, \tag{2}$$

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* See Pizzetti, loc. cit., § 2.

where W_u, W_v, W_z stand for the partial derivatives $\partial W/\partial u$, $\partial W/\partial v$, $\partial W/\partial z$ respectively. The equations of the plumb-bob locus of P_0 are

$$\frac{u-u_0}{W_u} = \frac{v-v_0}{W_v} = \frac{z-z_0}{W_z}. \tag{3}$$

From these equations it follows that P_0 (u_0, v_0, z_0) lies on this locus, and also that P_1 (u_1, v_1, z_1) lies on it, for since the straight line P_1P_0 is the vertical of P_1,

$$\frac{u_1-u_0}{W_u^{(1)}} = \frac{v_1-v_0}{W_v^{(1)}} = \frac{z_1-z_0}{W_z^{(1)}},$$

where $W_u^{(1)}$, $W_v^{(1)}$, $W_z^{(1)}$ denote the particular values of W_u, W_v, W_z respectively when $u = u_1$, $v = v_1$, $z = z_1$. It also follows from the systems of equations (2) and (3) that the plumb-bob locus of P_0 (represented by d in Fig. 1) has for tangent at P_0 the vertical P_0T of P_0.* Hence the

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* In order to prove this let us write equations (3) in the form

$$F = \begin{vmatrix} v-v_0 & z-z_0 \\ W_v & W_z \end{vmatrix} = 0, \quad \Phi = \begin{vmatrix} z-z_0 & u-u_0 \\ W_z & W_u \end{vmatrix} = 0.$$

Then for points of this curve

$$du : dv : dz = \begin{vmatrix} F_v & F_z \\ \Phi_v & \Phi_z \end{vmatrix} : \begin{vmatrix} F_z & F_u \\ \Phi_z & \Phi_u \end{vmatrix} : \begin{vmatrix} F_u & F_v \\ \Phi_u & \Phi_v \end{vmatrix}.$$

But

$$F_u = \begin{vmatrix} v-v_0 & z-z_0 \\ W_{vu} & W_{zu} \end{vmatrix}, \quad F_v = \begin{vmatrix} v-v_0 & z-z_0 \\ W_{vv} & W_{zv} \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ W_v & W_z \end{vmatrix},$$

$$F_z = \begin{vmatrix} v-v_0 & z-z_0 \\ W_{vz} & W_{zz} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ W_v & W_z \end{vmatrix},$$

$$\Phi_u = \begin{vmatrix} z-z_0 & u-u_0 \\ W_{zu} & W_{uu} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ W_z & W_u \end{vmatrix}, \quad \Phi_v = \begin{vmatrix} z-z_0 & u-u_0 \\ W_{zv} & W_{uv} \end{vmatrix},$$

$$\Phi_z = \begin{vmatrix} z-z_0 & u-u_0 \\ W_{zz} & W_{uz} \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ W_z & W_u \end{vmatrix}.$$

For the particular point P_0, (u_0, v_0, z_0) these derivatives have the values

$$F_u^{(0)} = 0, \qquad F_v^{(0)} = W_z^{(0)}, \quad F_z^{(0)} = -W_v^{(0)},$$

$$\Phi_u^{(0)} = -W_z^{(0)}, \quad \Phi_v^{(0)} = 0, \qquad \Phi_z^{(0)} = W_u^0.$$

Hence at P_0

$$du : dv : dz = W_z^{(0)}W_u^{(0)} : W_z^0W_v^0 : W_z^0W_z^0 = W_u^0 : W_v^0 : W_z^0.$$

It is assumed that the derivatives W_u, W_v, W_z do not all vanish at the same point (u, v, z) and that each of them has continuous first partial derivatives.

THEOREM 1. *The plumb-bob locus d of P_0 passes through the points P_0 and P_1 and is tangent at P_0 to the vertical P_0T of P_0* (see Fig. 1).

If we denote time by t, the differential equations of motion (with respect to the axes $O - u, v, z$) of the falling particle are*

$$\tag{4} \begin{gathered} \frac{d^2u}{dt^2} - 2\omega\frac{dv}{dt} - \omega^2 u = \frac{\partial V}{\partial u}, \\ \frac{d^2v}{dt^2} + 2\omega\frac{du}{dt} - \omega^2 v = \frac{\partial V}{\partial v}, \quad \frac{d^2z}{dt^2} = \frac{\partial V}{\partial z}. \end{gathered}$$

Since by relation (1)

$$\omega^2 u + \frac{\partial V}{\partial u} = \frac{\partial W}{\partial u}, \quad \omega^2 v + \frac{\partial V}{\partial v} = \frac{\partial W}{\partial v}, \quad \frac{\partial V}{\partial z} = \frac{\partial W}{\partial z},$$

equations (4) may be written in the form:†

$$\tag{5} \frac{d^2u}{dt^2} - 2\omega\frac{dv}{dt} = \frac{\partial W}{\partial u}, \quad \frac{d^2v}{dt^2} + 2\omega\frac{du}{dt} = \frac{\partial W}{\partial v}, \quad \frac{d^2z}{dt^2} = \frac{\partial W}{\partial z}.$$

The path c of the falling particle is that solution of equations (5) [or (4)] which is subject to the initial condition

$$\tag{6} \begin{gathered} \text{when } t = 0, \quad u = u_0, \quad v = v_0, \quad z = z_0, \\ \frac{du}{dt} = \frac{dv}{dt} = \frac{dz}{dt} = 0. \end{gathered}$$

Therefore it follows from equations (5) that

$$\tag{7} \left(\frac{d^2u}{dt^2}\right)_0 = W_u^{(0)}, \quad \left(\frac{d^2v}{dt^2}\right)_0 = W_v^{(0)}, \quad \left(\frac{d^2z}{dt^2}\right)_0 = W_z^{(0)},$$

where the subscript $_0$ indicates that those particular values of the derivatives d^2u/dt^2, d^2v/dt^2, d^2z/dt^2 have been taken for which $t = 0$. Since $W_u^{(0)}$, $W_v^{(0)}$, $W_z^{(0)}$ are proportional to the directional cosines of the vertical P_0T of P_0, we have, on account of relations (6) and (7), the

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* For a derivation of these equations see § 2 of the author's paper in the *Transactions Amer. Math. Society*, vol. 13, pp. 469–490, where x, y, z, U stand for u, v, z, V here used.

† A superficial reading of equations (5) might lead one *erroneously* to the conclusion that ω^2 was neglected. In these equations ω^2 is contained in the expressions of the derivatives $\partial W/\partial u$, $\partial W/\partial v$. In short, equations (5) are the equivalents of equations (4), and *not* the approximations thereto made by neglecting ω^2.

THEOREM 2. *The path c of the falling particle (with respect to the axes $O - u, v, z$) passes through the point P_0 and is tangent at P_0 to the vertical P_0T of P_0* (see Fig. 1).

§ 3. *Special Relations Corresponding to a Distribution of Revolution.*

By a *distribution of revolution* of the earth's gravitating matter we shall mean a distribution of matter for which the potential function (1) assumes the form

$$W = f(\tau, z), \tag{8}$$

where

$$\tau = \sqrt{u^2 + v^2}$$

represents the distance of the point (u, v, z) from the earth's axis of rotation Oz.

For such a distribution the level surfaces of the weight field are surfaces of revolution, the two-parameter family of lines of force of the weight field lie in the one-parameter family of planes which pass through the axis Oz, and the differential equation of those lines of force which lie in any such plane is

$$\frac{dz}{d\tau} = \frac{\partial f/\partial z}{\partial f/\partial \tau}; \tag{9}$$

the plumb-bob locus of P_0 lies in the plane of P_0 and Oz, and its equation in this plane is

$$\frac{\partial f}{\partial z}(\tau - \tau_0) - \frac{\partial f}{\partial \tau}(z - z_0) = 0. \tag{10}$$

The meridian planes of P_1 and P_0 then coincide and contain, besides the verticals of P_1 and P_0, the plumb-bob locus d of P_0. The path c of the falling particle does not lie in this plane. Let us project it orthographically on this plane and denote by c' and C' the projections of c and C respectively (see Fig. 2). Then it follows from Definition 2, § 1, that

$$\text{S.D.} = P_1C', \tag{11}$$

where S.D. stands for southerly deviation. The point T in which the vertical of P_0 pierces the horizontal plane of P_1 now (i. e., for a distribution of revolution) lies in the north-and-south line of P_1. Let us put (see Fig. 2)

$$\xi_1 = TC', \quad \xi_2 = TP_1,$$

the positive sense of each of these quantities being the same as that of S.D. Then

$$(12)\quad \xi_1 - \xi_2 = TC' - TP_1 = P_1T + TC' = P_1C' = \text{S.D.}$$

The curve c' is closely related to a curve c'' which we will now define. The path c of the falling particle may be regarded as the directrix of a surface of revolution of axis Oz. This surface of revolution passes through the point P_0. The intersection of this surface of revolution by the plane of P_0 and Oz is the curve c''.

We will now prove the two following theorems.

THEOREM 3. *The curvature, at P_0, of the plumb-bob locus d of P_0 is twice that of the line of force of the weight field which passes through P_0, and the concavity of these curves is in the same direction.*

THEOREM 4. *The curvature, at P_0, of the curves c' and c'' is the same in both magnitude and direction.*

In order to prove Theorem 3 let us note that, since the two curves are tangent, it will be sufficient to compare the values, which correspond to the two curves, of the second derivative $d^2z/d\tau^2$. For the line of force the expression for this second derivative is found by differentiating equation (9). Thus we find for the line of force that

$$(13)\quad \frac{d^2z}{d\tau^2} = \frac{\dfrac{\partial^2 f}{\partial\tau\partial z}\left\{\left(\dfrac{\partial f}{\partial\tau}\right)^2 - \left(\dfrac{\partial f}{\partial z}\right)^2\right\} + \dfrac{\partial f}{\partial\tau}\dfrac{\partial f}{\partial z}\left\{\dfrac{\partial^2 f}{\partial z^2} - \dfrac{\partial^2 f}{\partial\tau^2}\right\}}{\left(\dfrac{\partial f}{\partial\tau}\right)^3}.$$

Let us represent by $F(\tau, z)$ the left member of equation (10). The derivatives $dz/d\tau$, $d^2z/d\tau^2$ of the function defined by equation (10) are expressible in terms of the partial derivatives of F by means of the well known relations

$$(14)\quad \begin{aligned} \frac{\partial F}{\partial\tau} + \frac{\partial F}{\partial z}\cdot\frac{dz}{d\tau} &= 0,\\ \frac{\partial^2 F}{\partial\tau^2} + 2\frac{\partial^2 F}{\partial\tau\partial z}\cdot\frac{dz}{d\tau} + \frac{\partial^2 F}{\partial z^2}\left(\frac{dz}{d\tau}\right)^2 + \frac{\partial F}{\partial z}\cdot\frac{d^2z}{d\tau^2} &= 0. \end{aligned}$$

For the particular point $P_0(\tau_0, z_0)$ we have*

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* Since $F = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \dfrac{\partial f}{\partial\tau} & \dfrac{\partial f}{\partial z} \end{vmatrix}$,

$$\left(\frac{\partial F}{\partial \tau}\right)_0 = \left(\frac{\partial f}{\partial z}\right)_0, \quad \left(\frac{\partial F}{\partial z}\right)_0 = -\left(\frac{\partial f}{\partial \tau}\right)_0,$$

$$(15) \quad \left(\frac{\partial^2 F}{\partial \tau^2}\right)_0 = 2\left(\frac{\partial^2 f}{\partial \tau \partial z}\right)_0, \quad \left(\frac{\partial^2 F}{\partial \tau \partial z}\right)_0 = \left(\frac{\partial^2 f}{\partial z^2}\right)_0 - \left(\frac{\partial^2 f}{\partial \tau^2}\right)_0,$$

$$\left(\frac{\partial^2 F}{\partial z^2}\right)_0 = -2\left(\frac{\partial^2 f}{\partial \tau \partial z}\right)_0.$$

Hence for the plumb-bob locus we have, at P_0,

$$\left(\frac{dz}{d\tau}\right)_0 = -\frac{\left(\frac{\partial F}{\partial \tau}\right)_0}{\left(\frac{\partial F}{\partial z}\right)_0} = \frac{\left(\frac{\partial f}{\partial z}\right)_0}{\left(\frac{\partial f}{\partial \tau}\right)_0},$$

$$(16) \quad \left(\frac{d^2 z}{d\tau^2}\right)_0 = \frac{\left(\frac{\partial^2 F}{\partial \tau^2}\right)_0 + 2\left(\frac{\partial^2 F}{\partial \tau \partial z}\right)_0\left(\frac{dz}{d\tau}\right)_0 + \left(\frac{\partial^2 F}{\partial z^2}\right)_0\left(\frac{dz}{d\tau}\right)_0^2}{-\left(\frac{\partial F}{\partial z}\right)_0}$$

$$= 2\,\frac{\left(\frac{\partial^2 f}{\partial \tau \partial z}\right)_0\left\{\left(\frac{\partial f}{\partial \tau}\right)_0^2 - \left(\frac{\partial f}{\partial z}\right)_0^2\right\} + \left(\frac{\partial f}{\partial \tau}\right)_0\left(\frac{\partial f}{\partial z}\right)_0\left\{\left(\frac{\partial^2 f}{\partial z^2}\right)_0 - \left(\frac{\partial^2 f}{\partial \tau^2}\right)_0\right\}}{\left(\frac{\partial f}{\partial \tau}\right)_0^3}.$$

$$\frac{\partial F}{\partial \tau} = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \frac{\partial^2 f}{\partial \tau^2} & \frac{\partial^2 f}{\partial z \partial \tau} \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ \frac{\partial f}{\partial \tau} & \frac{\partial f}{\partial z} \end{vmatrix},$$

$$\frac{\partial F}{\partial z} = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \frac{\partial^2 f}{\partial \tau \partial z} & \frac{\partial^2 f}{\partial z^2} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ \frac{\partial f}{\partial \tau} & \frac{\partial f}{\partial z} \end{vmatrix},$$

$$\frac{\partial^2 F}{\partial \tau^2} = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \frac{\partial^3 f}{\partial \tau^3} & \frac{\partial^3 f}{\partial z \partial \tau^2} \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ \frac{\partial^2 f}{\partial \tau^2} & \frac{\partial^2 f}{\partial z \partial \tau} \end{vmatrix} + \frac{\partial^2 f}{\partial z \partial \tau},$$

$$\frac{\partial^2 F}{\partial \tau \partial z} = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \frac{\partial^3 f}{\partial^2 \tau \partial z} & \frac{\partial^3 f}{\partial z^2 \partial \tau} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ \frac{\partial^2 f}{\partial \tau^2} & \frac{\partial^2 f}{\partial z \partial \tau} \end{vmatrix} + \frac{\partial^2 f}{\partial z^2},$$

$$\frac{\partial^2 F}{\partial z^2} = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \frac{\partial^3 f}{\partial \tau \partial z^2} & \frac{\partial^3 f}{\partial z^3} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ \frac{\partial^2 f}{\partial \tau \partial z} & \frac{\partial^2 f}{\partial z^2} \end{vmatrix} - \frac{\partial^2 f}{\partial \tau \partial z}.$$

The first of these relations being the same as equation (9) for the point P_0 furnishes another proof of the fact, contained in Theorem 1, that the plumb-bob locus of P_0 is tangent, at P_0, to the line of force (of the weight field) which passes through P_0. This fact, taken in conjunction with the fact that the second of expressions (16) is + 2 times as great as expression (13) for the point P_0, constitutes the proof of Theorem 3.

In order to prove Theorem 4 let us express the equations of motion of the falling particle in terms of cylindrical coordinates. For this purpose let us put

$$u = \tau \cos\lambda, \quad v = \tau \sin\lambda, \quad z = z, \tag{17}$$

where $\tau = \sqrt{u^2 + v^2}$ has the same meaning as above, and λ is the angle which the plane through Oz and the general point (u, v, z) makes with the plane zOu. If we subject the system of equations (4) to this transformation, we obtain the equations

$$\begin{gathered}\frac{d^2\tau}{dt^2} - \tau\left(\omega + \frac{d\lambda}{dt}\right)^2 = \frac{\partial V}{\partial \tau}, \quad \frac{d}{dt}\left\{\tau^2\left(\omega + \frac{d\lambda}{dt}\right)\right\} = \frac{\partial V}{\partial \lambda}, \\ \frac{d^2 z}{dt^2} = \frac{\partial V}{\partial z}.\end{gathered} \tag{18}$$

By the same transformation relation (1) becomes

$$W = V + \frac{\omega^2}{2}\tau^2. \tag{19}$$

If we assume the potential function to be of the form (8),

$$\frac{\partial W}{\partial \lambda} = \frac{\partial V}{\partial \lambda} = 0, \tag{20}$$

and then the second of equations (18) yields the integral

$$\frac{d\lambda}{dt} + \omega = \frac{k}{\tau^2}, \tag{21}$$

where k is the constant of integration. The first of equations (18) then becomes

$$\frac{d^2\tau}{dt^2} = \frac{\partial V}{\partial \tau} + \frac{k^2}{\tau^3} = \frac{\partial f}{\partial \tau} - \omega^2\tau + \frac{k^2}{\tau^3}. \tag{22}$$

Furthermore, when $t = 0$ we have $d\lambda/dt = 0$ and $\tau = \tau_0$, whence by (21)

$$k = \omega\tau_0^2. \tag{23}$$

If now we put

$$\Omega(\tau, z) = f(\tau, z) - \frac{\omega^2}{2}\left(\tau^2 + \frac{\tau_0^4}{\tau^2}\right), \tag{24}$$

the equations of motion (for a distribution of revolution) take the form

$$\frac{d^2\tau}{dt^2} = \frac{\partial\Omega}{\partial\tau}, \quad \frac{d^2z}{dt^2} = \frac{\partial\Omega}{\partial z}, \quad \frac{d\lambda}{dt} = \omega\left(\frac{\tau_0^2}{\tau^2} - 1\right). \tag{25}$$

The path c of the falling particle is (for a distribution of revolution) that solution of equations (25) which is subject to the initial conditions

$$\begin{gathered} \text{when } t = 0, \quad \tau = \tau_0 = \sqrt{u_0^2 + v_0^2}, \qquad z = z_0, \\ \lambda = \lambda_0 = \operatorname{arc\,tan}\frac{v_0}{u_0}, \quad \frac{dz}{dt} = \frac{d\tau}{dt} = \frac{d\lambda}{dt} = 0. \end{gathered} \tag{26}$$

The curve c'' is now easily seen to be that solution of the pair of differential equations

$$\frac{d^2\tau}{dt^2} = \frac{\partial\Omega}{\partial\tau}, \quad \frac{d^2z}{dt^2} = \frac{\partial\Omega}{\partial z} \tag{27}$$

which is subject to the initial conditions

$$\text{when } t = 0, \quad \tau = \tau_0, \quad z = z_0, \quad \frac{d\tau}{dt} = \frac{dz}{dt} = 0. \tag{28}$$

For the same ordinate z the abscissas of the two curves c'' and c' differ by the amount

$$\begin{aligned} &\tau - \tau\cos(\lambda - \lambda_0) \\ &\qquad = 2\tau\sin^2\tfrac{1}{2}(\lambda - \lambda_0) = \tau[\tfrac{1}{2}(\lambda - \lambda_0)^2 + \cdots], \end{aligned} \tag{29}$$

where z and τ are those solutions of equations (27) which are subject to the conditions (28), or where z, τ, λ are those solutions of equations (25) which are subject to the conditions (26).

The solutions τ, z to terms of order not higher than the second in t are easily seen to be

$$\tau = \tau_0 + \alpha_2 t^2 + \cdots, \quad z = z_0 + \beta_2 t^2 + \cdots, \tag{30}$$

where

$$\alpha_2 = \frac{1}{2}\left(\frac{\partial\Omega}{\partial\tau}\right)_0, \quad \beta_2 = \frac{1}{2}\left(\frac{\partial\Omega}{\partial z}\right)_0,$$

the subscript $_0$ indicating that the values of the derivatives to which it is attached correspond to $\tau = \tau_0$ and $z = z_0$. Hence

$$\frac{\tau_0^2}{\tau^2} = 1 - \frac{2\alpha_2}{\tau_0}t^2 + \cdots,$$

and therefore, by the last of equations (25),

$$(31) \qquad \frac{d\lambda}{dt} = \omega\left(-\frac{2\alpha_2}{\tau_0}t^2 + \cdots\right),$$

whence, by (26),

$$(32) \qquad \lambda - \lambda_0 = -\tfrac{2}{3}\omega\frac{\alpha_2}{\tau_0}t^3 + \cdots.$$

Therefore the difference (29) becomes

$$(33) \qquad \begin{aligned} \tau - \tau\cos(\lambda - \lambda_0) &= (\tau_0 + \alpha_2 t^2 + \cdots)\left(\frac{2}{9}\frac{\alpha_2^2\omega^2}{\tau_0^2}t^6 + \cdots\right) \\ &= \frac{2}{9}\frac{\alpha_2^2\omega^2}{\tau_0}t^6 + \cdots \\ &= \frac{2}{9}\frac{\omega^2\alpha_2^2}{\tau_0\beta_2^3}(z - z_0)^3 + \cdots. \end{aligned}$$

Since this difference is of the third order in $(z - z_0)$ it follows that the curves c' and c'' have contact of the second order at the point P_0 and hence we have proved Theorem 4.

§ 4. *Some Known Theorems.*

The following three theorems will be found useful:

THEOREM 5. *In a two-dimensional field of force, for which a potential function exists, the curvature of a line of force is the derivative of the logarithm of the force taken in that direction, on an equipotential curve, in which the force increases.*

THEOREM 6. *If, in a positional field of force, a particle starts from rest, the initial curvature of the path described is one-third of the curvature of the line of force through the initial position.**

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* See Kasner's Princeton Colloquium Lectures, p. 9, footnote 2.

THEOREM 7. *If at a point P of a curve (with a continuously turning tangent) at which the curvature is finite (i. e., neither zero nor infinite) the tangent and the inner normal be taken as a set of rectangular axes, then the equation of the curve referred to these axes may be written in the form*

$$y = \frac{1}{2} \cdot \frac{1}{\rho} x^2 + \text{an expression of order three, or higher, in } x,$$

where x is measured from P along the tangent, y is measured from P along the inner normal and ρ is the radius of curvature at P.[*]

References to proofs of Theorems 6 and 7 are given in the following footnotes. In order to prove Theorem 5 let us refer the field of force to a set of rectangular axes (x, y) and denote by X, Y the components of the force j of the field. Then the differential equation of the lines of force is

$$\frac{dy}{dx} = \frac{Y}{X}, \tag{34}$$

whence

$$\frac{d^2y}{dx^2} = \frac{X^2\dfrac{\partial Y}{\partial x} - Y^2\dfrac{\partial X}{\partial y} + XY\left(\dfrac{\partial Y}{\partial y} - \dfrac{\partial X}{\partial x}\right)}{X^3}. \tag{35}$$

On the other hand, the magnitude of the force is

$$j = \sqrt{X^2 + Y^2} = X \cos \phi + Y \sin \phi, \tag{36}$$

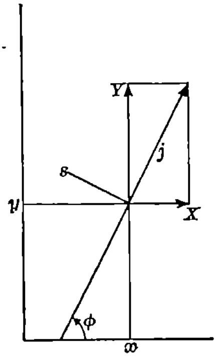

FIG. 3.

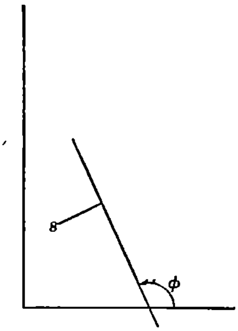

FIG. 4.

where $\tan \phi$ is the slope of the vector representing the force

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[*] See Osgood's Introduction to Infinite Series, p. 39. Also Goursat-Hedrick, Mathematical Analysis, vol. 1, p. 451.

(Fig. 3). Hence

$$0 = -X\sin\phi + Y\cos\phi,$$

whence

$$j\cos\phi = X, \quad j\sin\phi = Y.$$

Then

$$\frac{\partial j}{\partial x} = \frac{1}{j}\left(X\frac{\partial X}{\partial x} + Y\frac{\partial Y}{\partial x}\right), \quad \frac{\partial j}{\partial y} = \frac{1}{j}\left(X\frac{\partial X}{\partial y} + Y\frac{\partial Y}{\partial y}\right).$$

If now we represent by s (Figs. 3 and 4) distance measured on a line perpendicular to the vector representing j and such that s is positive on the side opposite to that on which the angle ϕ lies, then

$$\text{(37)} \quad \begin{aligned} \frac{\partial j}{\partial s} &= -\frac{\partial j}{\partial x}\sin\phi + \frac{\partial j}{\partial y}\cos\phi = \frac{1}{j}\left(-\frac{\partial j}{\partial x}Y + \frac{\partial j}{\partial y}X\right) \\ &= \frac{X^2\frac{\partial X}{\partial y} - Y^2\frac{\partial Y}{\partial x} + XY\left(\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial x}\right)}{j^2}. \end{aligned}$$

If now

$$\text{(38)} \quad \frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x},$$

which is the case when a potential function exists, then by (35) and (37),

$$\text{(39)} \quad \frac{d^2y}{dx^2} = \frac{j^2}{X^3}\cdot\frac{\partial j}{\partial s},$$

whence

$$\text{(40)} \quad \frac{\partial}{\partial s}\log_e j = \frac{\frac{\partial j}{\partial s}}{j} = \cos^3\phi\,\frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}},$$

since

$$\cos\phi = \frac{1}{\sqrt{1 + \tan^2\phi}} = \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}.$$

Hence Theorem 5.

§ 5. *Applications of the Preceding Theorems to the Problem under Consideration.*

Let us now replace the variables x and y by τ and z respectively, and denote by U the potential function of the field of force of Theorem 5. Then

$$(41)\qquad j\cos\phi = X = -\frac{\partial U}{\partial \tau},\quad j\sin\phi = Y = -\frac{\partial U}{\partial z},$$

$$j = \sqrt{\left(\frac{\partial U}{\partial \tau}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2},$$

and relation (37) becomes

$$(42)\qquad \frac{\frac{\partial j}{\partial s}}{j} = \frac{\Theta(U)}{j^3},$$

where

$$\Theta(U) = \frac{\partial^2 U}{\partial \tau \partial z}\left[\left(\frac{\partial U}{\partial z}\right)^2 - \left(\frac{\partial U}{\partial \tau}\right)^2\right] + \frac{\partial U}{\partial \tau}\cdot\frac{\partial U}{\partial z}\left[\frac{\partial^2 U}{\partial \tau^2} - \frac{\partial^2 U}{\partial z^2}\right].$$

If now $U = f(\tau, z)$ is the function (8),

$$(43)\qquad j = g = \sqrt{\left(\frac{\partial f}{\partial \tau}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2},$$

where g is the acceleration due to weight, formula (42) becomes

$$(44)\qquad \frac{\frac{\partial g}{\partial s}}{g} = \frac{\Theta(f)}{g^3}.$$

If, further, we denote by ξ distance measured along the north-and-south line of P_0 (positive to the south), we obtain the relation

$$(45)\qquad -\frac{\left(\frac{\partial g}{\partial \xi}\right)_0}{g_0} = \frac{[\Theta(f)]_0}{g_0^3},$$

in which, by Theorem 5 or relation (40), $-(\partial g/\partial \xi)_0/g_0$ is the curvature at P_0 of the line of force of the weight field which passes through P_0. Hence by Theorem 3 we have

THEOREM 8. *The curvature at P_0 of the plumb-bob locus d of P_0 is*

$$-2\frac{\left(\frac{\partial g}{\partial \xi}\right)_0}{g_0}. \tag{46}$$

Let us next put $U = \Omega$, where Ω is the function defined in (24). By (24)

$$\begin{gathered}\frac{\partial \Omega}{\partial \tau} = \frac{\partial f}{\partial \tau} - \omega^2\left(\tau - \frac{\tau_0^4}{\tau^3}\right), \quad \frac{\partial \Omega}{\partial z} = \frac{\partial f}{\partial z}, \\ \frac{\partial^2 \Omega}{\partial \tau^2} = \frac{\partial^2 f}{\partial \tau^2} - \omega^2\left(1 + 3\frac{\tau_0^4}{\tau^4}\right), \quad \frac{\partial^2 \Omega}{\partial \tau \partial z} = \frac{\partial^2 f}{\partial \tau \partial z}, \quad \frac{\partial^2 \Omega}{\partial z^2} = \frac{\partial^2 f}{\partial z^2},\end{gathered} \tag{47}$$

where f is the function (8). For the particular point P_0, whose coordinates are τ_0 and z_0, these derivatives assume the values

$$\begin{gathered}\left(\frac{\partial \Omega}{\partial \tau}\right)_0 = \left(\frac{\partial f}{\partial \tau}\right)_0, \quad \left(\frac{\partial \Omega}{\partial z}\right)_0 = \left(\frac{\partial f}{\partial z}\right)_0, \\ \left(\frac{\partial^2 \Omega}{\partial \tau^2}\right)_0 = \left(\frac{\partial^2 f}{\partial \tau^2}\right)_0 - 4\omega^2, \quad \left(\frac{\partial^2 \Omega}{\partial \tau \partial z}\right)_0 = \left(\frac{\partial^2 f}{\partial \tau \partial z}\right)_0, \\ \left(\frac{\partial^2 \Omega}{\partial z^2}\right)_0 = \left(\frac{\partial^2 f}{\partial z^2}\right)_0,\end{gathered} \tag{48}$$

whence

$$j_0 = \sqrt{\left(\frac{\partial f}{\partial \tau}\right)_0^2 + \left(\frac{\partial f}{\partial z}\right)_0^2} = g_0,$$

$$\begin{aligned}[\Theta(\Omega)]_0 &= [\Theta(f)]_0 - 4\omega^2\left(\frac{\partial f}{\partial \tau}\right)_0\left(\frac{\partial f}{\partial z}\right)_0 \\ &= [\Theta(f)]_0 - 4\omega^2 g_0^2 \sin\phi_0 \cos\phi_0.\end{aligned} \tag{49}$$

Therefore for the particular function Ω and the particular point P_0 we have, by (42),

$$\begin{aligned}\frac{\left(\frac{\partial j}{\partial s}\right)_0}{j_0} &= \frac{[\Theta(f)]_0}{g_0^3} - \frac{4\omega^2 \sin\phi_0 \cos\phi_0}{g_0} \\ &= -\frac{\left(\frac{\partial g}{\partial \xi}\right)_0}{g_0} - \frac{4\omega^2 \sin\phi_0 \cos\phi_0}{g_0}, \quad \text{by (45).}\end{aligned} \tag{50}$$

By Theorem 5 or relation (40) this expression (50) is the curvature at P_0 of the line of force, passing through P_0, of the positional field of force of potential function Ω. Hence by Theorems 4 and 6 we have

THEOREM 9. *The curvature at P_0 of the orthographic projection c' on the meridian plane of P_0 of the path of the falling particle is*

$$-\frac{1}{3}\frac{1}{g_0}\left[\left(\frac{\partial g}{\partial \xi}\right)_0 + 4\omega^2 \sin\phi_0 \cos\phi_0\right]. \tag{51}$$

We have now found expressions for the curvature at P_0 of the curves d and c', which pass through the point P_0, lie in the meridian plane of P_0, and have (by Theorems 1 and 2) as common tangent at P_0 the vertical P_0T of P_0 (Fig. 2). By Theorem 7 we can now find expressions for the quantities ξ_1 and ξ_2 which were defined between equations (11) and (12). For this purpose let us denote by h the height of fall P_1P_0. In the first place, let us drop perpendiculars from P_1 and C' to the line P_0T and denote by $\overline{P}_1$ and $\overline{C}'$ respectively, the feet of these perpendiculars (Fig. 5). Then the infinitesimals $P_0\overline{P}_1$, $P_0\overline{C}'$, $\overline{P}_1P_1$, $\overline{C}'C'$ differ from the infinitesimals $h = P_0P_1$, h, $\xi_2 = TP_1$, $\xi_1 = TC'$ respectively by infinitesimals of higher order. Therefore by Theorems 7, 8 and 9

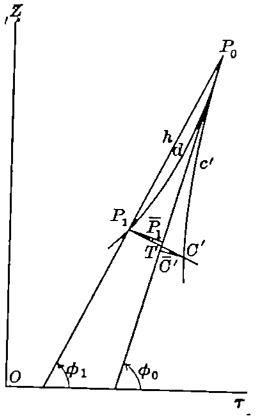

FIG. 5.

$$\xi_2 = \pm \frac{1}{2} \left\{ -2 \frac{\left(\dfrac{\partial g}{\partial \xi}\right)_0}{g_0} \right\} h^2 + \epsilon,$$

$$\xi_1 = \pm \frac{1}{2} \left\{ -\frac{1}{3g_0} \left[\left(\frac{\partial g}{\partial \xi}\right)_0 + 4\omega^2 \sin \phi_0 \cos \phi_0 \right] \right\} h^2 + \epsilon',$$

where ϵ and ϵ' are infinitesimals of order higher than that of h^2. In order to remove the ambiguity in sign let us observe that since the latitute ϕ lies between $-90°$ and $+90°$, it follows that $\cos \phi > 0$, and therefore, by relation (40), $(\partial j/\partial s)/j$ and d^2y/dx^2 are either both positive or both negative. Hence it follows from (45) and Theorem 8 that for the curve d, $-(\partial g/\partial \xi)_0/g_0$ and $(d^2z/d\tau^2)_0$ are either both positive or both negative; and from (50) and Theorem 9 that for the curve c', $-1/g_0\{(\partial g/\partial \xi)_0 + 2\omega^2 \sin 2\phi_0\}$ and $(d^2z/d\tau^2)_0$ are either both positive or both negative. Furthermore, we assumed ξ_1 and ξ_2 to be positive toward the equator. Consequently if for definiteness we suppose (as shown in Fig. 5) that for curve d, $(d^2z/d\tau^2)_0 > 0$ and for curve c', $(d^2z/d\tau^2)_0 < 0$, it follows that $\xi_2 = -P_1T < 0$ and $\xi_1 = TC' > 0$. Therefore in the above expressions for ξ_2 and ξ_1 the lower signs must be used and thus we have

$$\xi_2 = \frac{\left(\dfrac{\partial g}{\partial \xi}\right)_0}{g_0} h^2, \tag{52}$$

$$\xi_1 = \left\{ 2\omega^2 \sin 2\phi_0 + \left(\frac{\partial g}{\partial \xi}\right)_0 \right\} \frac{h^2}{6g_0}, \tag{53}$$

to terms of order not higher than the second in h; whence

$$\text{S.D.} = \xi_1 - \xi_2 = \left\{ 2\omega^2 \sin 2\phi_0 - 5 \left(\frac{\partial g}{\partial \xi}\right)_0 \right\} \frac{h^2}{6g_0}, \tag{54}$$

which is formula (I).

NOTE ON SOLVABLE QUINTICS.

BY PROFESSOR F. N. COLE.

(Read before the American Mathematical Society, January 2, 1915.)

THE substance of the following paper was included several years ago in my university lecture course on the theory of groups. Being laid away for seasoning and final touches, it was destroyed in the fire of last October, along with the rather lengthy computations under Lagrange's theorem involved in the transition from the equations of Runge to those here employed. I have succeeded in restoring the paper from the lecture notes of my students, and am moved to publish it now in order that whatever value it may have may not be further exposed to destruction. The process of solution seems to be reduced to the simplest terms, and I have not been able to find that it has been anticipated.

The quintic is supposed to have been reduced (by Tschirnhausen transformation) to the form $x^5 + \alpha x + \beta = 0$. In his paper in the *Acta Mathematica*, volume 7 (1885), Runge obtained the sextic resolvent

$$(v - \alpha)^4(v^2 - 6\alpha v + 25\alpha^2) = 5^5\beta^4 v,$$

where $v = (x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1)^2$, the x's being the roots of the quintic, and expressed the coefficients of solvable (irreducible) quintics in terms of two parameters. The condition for solvability is of course that the resolvent sextic shall have a rational root. The function v belongs to the group of order 20, and the problem is substantially to calculate by radicals the four Lagrange functions

$$\tau_1 = x_1 + \epsilon x_2 + \epsilon^2 x_3 + \epsilon^3 x_4 + \epsilon^4 x_5, \qquad \tau_3 = x_1 + \epsilon^3 x_2 + \epsilon x_3 + \epsilon^4 x_4 + \epsilon^2 x_5,$$

$$\tau_2 = x_1 + \epsilon^2 x_2 + \epsilon^4 x_3 + \epsilon x_4 + \epsilon^3 x_5, \qquad \tau_4 = x_1 + \epsilon^4 x_2 + \epsilon^3 x_3 + \epsilon^2 x_4 + \epsilon x_5.$$

These last belong to the identical group, their fifth powers to a group of order 5, the combinations $\sigma_1 = \tau_1^5 + \tau_4^5$, $\sigma_2 = \tau_2^5 + \tau_3^5$ to a group of order 10, and finally $5^3\rho = \sigma_1 + \sigma_2$ to the same group of order 20 as Runge's v.

I find now the following series of resolvent equations:

$$(1) \qquad 4^5\alpha^5(\rho^2 + 6\beta\rho + 25\beta^2) = (\rho + 25\beta)(\rho + 5\beta)^5;$$

$$(2) \quad \sigma_1 + \sigma_2 = 5^3\rho, \quad \sigma_1\sigma_2 = -\frac{5^6}{2^4}(\rho^2 + 30\beta\rho + 125\beta^2);$$

$$(3) \quad \begin{matrix} \tau_1^5 + \tau_4^5 = \sigma_1, \\ \tau_2^5 + \tau_3^5 = \sigma_2. \end{matrix} \quad \tau_1^5\tau_4^5 = \frac{5^6}{2^6}\frac{(\rho + 25\beta)^3}{\sigma_2 - \sigma_1} = -\tau_2^5\tau_3^5.$$

As an example, the equation $x^5 + 15x + 12 = 0$ gives $\rho = 60$, $\sigma_1 + \sigma_2 = 5^4 \cdot 12$, $\sigma_1\sigma_2 = -5^8 \cdot 108$, $\sigma_1 = -5^4 \cdot 6$,

$\sigma_2 = 5^4 \cdot 18.$ $\tau_1^5 + \tau_4^5 = -5^4 \cdot 6,$ $\tau_1^5\tau_4^5 = 3^5 \cdot 5^5,$ $\tau_1^5 = -1875 + 525\sqrt{10},$ $\tau_4^5 = -1875 - 525\sqrt{10}.$ $\tau_2^5 + \tau_3^5 = 5^4 \cdot 18,$ $\tau_2^5\tau_3^5 = -3^5 \cdot 5^5,$ $\tau_2^5 = 5625 + 1800\sqrt{10},$ $\tau_3^5 = 5625 - 1800\sqrt{10}.$ $5x = \tau_1 + \tau_2 + \tau_3 + \tau_4.$

The connection with Runge's resolvent is effected by the relation

$$\rho = -5\beta\frac{v - 5\alpha}{v - \alpha},$$

by which equation (1) may be verified. The relation

$$\sigma = \frac{-5^4\beta}{2(v - \alpha)}(v - 5\alpha + \sqrt{v^2 - 6\alpha v + 25\alpha^2}),$$

which includes the preceding and gives the key to equations (2) and (3), was worked out by Lagrange's theorem.

COLUMBIA UNIVERSITY,
April 6, 1915.

THE MADISON COLLOQUIUM LECTURES ON MATHEMATICS.

Part I: On Invariants and the Theory of Numbers. By LEONARD EUGENE DICKSON. New York, American Mathematical Society, 1914.

THE number of new mathematical systems which may be characterized as distinct mutations, whose discovery or development is to be credited to American research, has shown a marked increase within a few decades. The reviewer of Professor Dickson's Lectures of the Madison Colloquium volume has the satisfaction of recording one of these great discoveries, his theory of classes in invariant theory, and of observing how as a result of this discovery, number theory, which long had little contact with the theory of invariants, now has very much in common with it. Dickson's technical memoirs in which the theory of classes and the invariant theory of modular forms were first expounded appeared in 1909. And while the material and indeed much of the method also of the Colloquium Lectures are new, they are dominated by the theory of classes and may, therefore, be regarded as a superstructure of the system founded in his 1909 papers. Lecture I may be regarded, also, as introductory to the theory as a whole.

The theory of classes is a general invariant theory, applicable, and applied by the lecturer, to three types of invariants. We now define these. Ordinary invariant theory of algebraical quantics, which we may call the *algebraic* theory, is one in which the coefficients of the linear transformations as well as the coefficients of the transformed quantics themselves are perfectly arbitrary variables. If both of these sets of coefficients are parameters representing residues of a prime number p (or, generally, marks of a Galois field), we have the invariant theory called *modular* invariant theory, due to Dickson. Again if the coefficients of the quantics are variables while the coefficients of the transformations are modular the invariant formations are called *formal modular*. This type of invariant was first defined by A. Hurwitz.

Suppose that S is a system of modular forms, and to follow the inductive plan of the Lectures, suppose that the modulus is p and that S consists of one modular quadratic form q_m in m variables. The coefficients β_{ij} of q_m are parameters to which may be assigned in turn particular sets of residues modulo p, giving the particular forms q_m', q_m'', $\cdots$. Now if the totality of forms $q_m^{(k)}$ be transformed by all of the transformations of the linear group L (mod p) on the variables, the $q_m^{(k)}$ are separated into classes C_i such that two particular forms belong to the same class if and only if they are equivalent under L.

The definition of an invariant now becomes a function-theoretic matter. A single-valued function φ of the undetermined coefficients β_{ij} is an invariant of q_m if φ has the same value for all sets β_{ij}', β_{ij}'', $\cdots$ of coefficients of forms q_m', q_m'', $\cdots$ belonging to the same class.

To determine the value of an invariant φ for a given class C_i we need only assign to the β_{ij} in φ the particular coefficients in a canonical form of q_m which belongs to C_i. In consequence invariants may here be determined from their values by an interpolation formula, or by some particular method whose use would be equivalent to the determination of a function by the interpolation process. And to determine a fundamental system of invariants we need only to determine a set which completely characterizes the classes C_i. Specifically a set of invariants φ_1, φ_2, $\cdots$ is said to characterize completely the classes when each φ_k has the same value for two classes only when the latter are identical, and the following theorem proved in Lecture I, taken with the interpolation idea, furnishes not only a powerful construction method, but also an advantageous means of proving that a system which has been constructed forms a fundamental system: If the modular invariants A, B, $\cdots$, J completely characterize the classes, they form a fundamental system of modular invariants.

The particular problems treated in Lecture I are, first, the reduction of the algebraic quadratic form

$$q_m = \sum_{i,j=1}^{m} \beta_{ij} x_i x_j \qquad (\beta_{ij} = \beta_{ji})$$

to the canonical form

$$x_1^2 + \cdots + x_{m-1}^2 + D x_m^2 \qquad (D = |\beta_{ij}|), \tag{1}$$

or to

$$x_1^2 + \cdots + x_r^2, \tag{2}$$

according as the rank r of D is $= m$ or $< m$. This can be done by linear transformations with complex coefficients of determinant unity, and thus all algebraic quadratic forms may be separated into the classes

$$C_{m,D},\ C_r \quad (D \neq 0,\ r = 0, 1, \cdots, m-1),$$

where, for a particular value of D, $C_{m,D}$ is composed of all forms q_m of determinant D, each being transformable into (1), and so on. Every single-valued algebraic invariant of q_m is a single-valued function of D and r which completely characterize the classes.

Secondly, the corresponding canonical reduction of the modular q_m is made; the forms are (1), (2) and

$$x_1^2 + \cdots + x_{r-1}^2 + \rho x_r^2 \qquad (0 < r < m), \tag{3}$$

and the classes

$$C_{m,D},\ C_{r,+1},\ C_{r,-1},\ C_0 \quad (D=1, \cdots, p-1;\ r=1, \cdots, m-1),$$

where, for instance, $C_{r,-1}$ is composed of all forms transformable into (3). A fundamental system of rational integral modular invariants of q_m is then

$$D,\ A_1,\ \cdots,\ A_{m-1},\ I_0,$$

where if $M_1, M_2, \cdots, M_n$ denote the principal minors of order r of D, and d ranges over the principal minors of orders $> r$,

$$A_r = \{M_1^{(p-1)/2} + M_2^{(p-1)/2}(1 - M_1^{p-1}) + \cdots$$

$$+ M_n^{(p-1)/2}(1 - M_1^{p-1}) \cdots (1 - M_{n-1}^{p-1})\} \times \Pi(1 - d^{p-1}).$$

This invariant has the value $+1$ for any form of class $C_{r,1}$, for example. Also $I_0 = \Pi(1 - \beta_{ij}^{p-1})$.

In the opening paragraphs of Lecture II the algebraic binary quartic forms f are separated into classes by transformations of the type (algebraic) T: $x = x' + ty'$, $y = y'$. Since invariants under T are seminvariants of f we arrive at a determination of five seminvariants of f from the point of view of the classes, and a proof that they form a fundamental system of rational integral seminvariants. This set does not completely characterize the classes, i. e., is not a fundamental system of *single-valued* seminvariants of f. If f is modular and $p > 3$ eight determinate seminvariants characterize the classes.

Professor Dickson next establishes an inductive method of constructing all of the members of a fundamental system of modular seminvariants of a form of order n from the system for a form of order $n - 1$. For instance if n is divisible by p ($n = pq$), and $F_n = A_0x^n + A_1x^{n-1}y + \cdots$, the set consisting of A_0 and the fundamental system for

$$(4) \qquad \bar{F}_{n-1} = \frac{1}{y}(F_n - \varphi) \qquad [\varphi = A_0(x^p - xy^{p-1})^q]$$

completely characterize the classes of F_n.

By this and similar processes an explicit fundamental system of modular seminvariants of F_n ($p > n$) is constructed and for particular low orders some explicit systems for $p \ngtr n$. Invariants are then treated as seminvariants which possess the right type of symmetry, and the subject of linearly independent sets claims attention.

Lecture III is devoted to concomitants of the formal modular type. The first problem solved is that of the determination of a set of rational integral invariant functions of the variables of the modular transformations L, alone, such that any other such function is a rational integral function of those determined, with integral coefficients; i. e., the determination of a fundamental system of universal covariants of the group L. The first such determination was made by Dickson in a paper published in 1911. But for the case of two variables the work is here simplified by the introduction of geometrical concepts.

The transformations being

$$(5)\quad G\colon x' \equiv bx + dy, \quad y' \equiv cx + ey, \quad be - cd \equiv 1 \pmod{p},$$

and a point being defined as a pair of homogeneous coordinates $(x, y) = (kx, ky)$, we say that a point is a special point if it is invariant under at least one transformation which is not identity. For it

$$x' \equiv \rho x, \quad y' \equiv \rho y,$$

and ρ is a root of the characteristic congruence

$$(6)\qquad \rho^2 - (b + e)\rho + 1 \equiv 0 \pmod{p}.$$

Only real special points are invariant when (6) has an integral root, and all real points are conjugate under G. It follows that if an invariant of G vanishes for one of the real points it vanishes for all and has the factor

$$L = y\prod_{a=0}^{p-1}(x - ay) \equiv x^p y - xy^p \pmod{p}.$$

If (6) has Galois imaginary roots, the corresponding invariant, representing the conjugate set of imaginary special points, is

$$Q = (x^{p^2}y - xy^{p^2}) \div L,$$

and L, Q is the system sought.

The details of the proof here contain the two principal elements in the author's main method of constructing formal modular invariants and showing what ones are reducible. For with $p = 2$ and $f = ax^2 + bxy + cy^2$ the transformation $x = x' + y'$, $y = y'$ induces the transformation

$$a' \equiv a, \quad b' \equiv b, \quad c' \equiv a + b + c \pmod{2}.$$

This latter may be identified with a special case of (5), and the two universal covariants of this special group become formal modular seminvariants of f. As to reducibility, in view of the theory of conjugate points, we need only show that a covariant has the factor y in order to know that it has the factor L.

Fundamental systems of formal modular seminvariants and invariants of the binary quadratic form modulo p are completely determined in this Lecture, as well as sets for the cubic, and some simpler modular covariant systems modulo 2.

Brief mention is given to the total binary group's form problem, and the invariantive classification of forms.

We now come to the two lectures on modular geometry, and in Lecture IV, the modular geometry and covariantive theory of a quadratic form in m variables modulo 2. Modular geometry is not a new term, but Dickson's formulation of a new theory under the old name marks a notable advance. In modular theories such as G. Arnoux's Essai de Géométrie analytique modulaire (1911), a "curve" is a finite aggregate of real points. Thus an analytic representative of this curve, its quantic, is not determinate in any sense from its points. Dickson assumes at the beginning that the curve, represented for instance by the modular form

$$q_m(x) = \Sigma c_{ij}x_ix_j + \Sigma b_ix_i^2 \quad (i, j = 1, \cdots, m; i < j),$$

modulo 2, shall contain an infinitude of points. That is, he defines a point as a set of m ordered elements $(x_1, \cdots, x_m)$, not all zero, of the infinite field F_2 composed of the roots of all congruences modulo 2 with integral coefficients. The point $(x) = (x_1, \cdots, x_m)$ is called real if the ratios of the x's are congruent to integers modulo 2, otherwise it is imaginary. Then the aggregate of points (x) for which $q_m(x) \equiv 0$ (mod 2) is called a quadric locus, a conic if $m = 3$. The quadric is thus composed of an infinitude of points, only a finite number of which are real. The investigation of the modular invariant theory of this locus is carried out as a purely arithmetical theory, without any geometrical representation of the locus, although the terminology and to some extent the methods of analytic projective geometry are employed. The lack of any mode of geometrical representation leaves the reader with a feeling of conjecture as to just what kind of geometry in the concrete he is here concerned with, and as to whether something similar, perhaps, to isometric projection could be invented to give a picture of the infinite point cluster constituting the modular curve.

The first invariant formation treated in this lecture is the polar locus

$$P(y, z) \equiv \Sigma c_{ij}(y_iz_j + y_jz_i) \text{ (mod 2)}.$$

For an odd m the polars of all points (y) have at least one point in common. A determinate common point, whose coordinates happen to be cogredient to the variables, is called the apex of the quadric. Any line through the apex is tangent to the quadric, and conversely. Thus the quadric has a linear tangential equation. If (C) is the apex, $q_m(C)$ is a formal modular invariant. If it vanishes, the apex is on the locus, which is then a cone. In the case of a conic, reducible to

$$\varphi \equiv X_1X_2 + X_3^2,$$

the only real points on the locus are (1, 1, 1), (1, 0, 0), (0, 1, 0). The apex is (0, 0, 1). The only other real points, three in number, lie on the covariant line

$$X_1 + X_2 + X_3 \equiv 0 \pmod{2}. \tag{7}$$

A like configuration on the real points defined by the quinary surface, of great beauty, is constructed, and a similar theory for the case of an even m is given.

The latter half of this chapter, the most technical part of the Lectures, is devoted to a determination, from the standpoint of the classes, of a fundamental system of modular covariants of the ternary quadratic form F with integral coefficients modulo 2.

With a modular analytic projective geometry defined and its covariant theory established Professor Dickson proceeds in Lecture V to a particular curve and a particular feature of the geometry on that curve. This is a theory of plane cubic curves with a real inflexion point which holds true both in ordinary and in modular geometry. After reducing the cubic to the normal form

$$C = x^2y + gy^3 + hy^2z + \delta z^3 \qquad (\delta \neq 0),$$

the author develops the theory of the inflexions. A prominent part is played by the invariants

$$s = -3\delta h, \quad t = -108\delta^2 g.$$

A cubic with integral coefficients taken modulo p, a prime > 3, with at least one real inflexion point and with invariant $s = 0$, and $t \neq 0$, has nine real inflexion points if p is of the form $3j + 1$ and $-t$ is a sixth power modulo p, a single real inflexion point if $p = 3j + 1$ and $-t$ is a quadratic non-residue of p, and exactly three real inflexion points in all other cases. The author proves a series of similar theorems.

As a whole these Lectures are indeed a most meritorious contribution, suggesting many new problems of many new kinds.

O. E. GLENN.

SOME BOOKS ON CALCULUS.

Elements of the Differential and Integral Calculus (Revised Edition). By W. A. GRANVILLE. Edited by P. F. SMITH. Boston, Ginn and Company, 1911. xv+463 pp.

Elementary Textbook on the Calculus. By V. SNYDER and J. I. HUTCHINSON. New York, American Book Company, 1912. 384 pp.

The Calculus. By E. W. DAVIS assisted by W. C. BRENKE. Edited by E. R. HEDRICK. New York, The Macmillan Company, 1913. xx+383+63 pp.

Esercizi di Analisi Infinitesimale. By G. VIVANTI. Pavia, Mattei, 1913. vii+470 pp.

GRANVILLE'S Calculus is too widely known both in its original and in its revised edition to require any long notice.* A number of changes have been introduced in the revision and all seem to improve the work as a class drill book. In the number of pages the additions and subtractions exactly balance.

In the preface the author states that in the last few years considerable progress had been made in the teaching of the elements of the calculus and in this revised edition the latest and best methods are exhibited. This statement is entirely incomprehensible to us. So far as we have observed the only important improvement in teaching calculus has been to introduce the calculus earlier in the student's course and so to present it in matter as in time that it may be of greater use to the student in his courses on physics and mechanics. Granville's book veers not the slightest toward this point, no more in the revised than in the original edition.

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* For a review of the original see E. B. Van Vleck, this BULLETIN, volume 12, pages 181–187. We are personally out of sympathy with that review because we believe that it represents the view-point of the mathematician catering to the one per cent of the students of calculus who will possibly be pure mathematicians rather than the point of view of the teacher of mathematics who sets his heart on doing the maximum good to the maximum number and who regards mathematics through calculus as essentially the handmaiden of the theoretical and applied sciences. We believe that the aristocratic movement has passed its zenith and is giving way to a less selfish and more democratic point of view,—and we daresay the earlier reviewer is in sympathy with the change.

The calculus is taught to such a large number of students in so many institutions that there is no particular reason why any teacher who has a prominent position cannot find or should not find a publisher for his own notes on calculus and thus have a text of his own which suits him better than any other. This is sufficient excuse for the appearance of Snyder and Hutchinson's book. The work is short. It could have been made shorter without harm by abridging the 42 pages given to contact and curvature, singular points, and envelopes. The most natural book with which to compare Snyder and Hutchinson's is Osborne's (revised edition, 1908). The two are a good deal alike; they give the calculus which is really needed and give it in direct teachable form,—which must be balm to the souls of those that are bored by the modernization of calculus toward rigor, or toward "practical mathematics," or toward the so-called "mixed method."

In their preface the authors call attention to the pressure toward shortening the course in mathematics, they cite the appearance of books on calculus for engineers, physicists, chemists, and so on, and state that it is in recognition of this pressure that they have written. It is good that they are alive to the advisability of adapting calculus to the students who take it; we should all be alive to that fundamental principle of educational justice. But is there any real pressure to shorten the course in mathematics? Is not the pressure rather to get the kind of mathematics the student, in the opinion of engineers, etc., needs? And there is plenty of that kind. Is not the shortening merely an indirect result due to the fact that we will not give the student that which others think he needs and that they therefore diminish his time with us so that they may give him what, in their opinion, he needs more than what we would offer him in any additional time allowed to us?

We may quote from the introduction to Perry's Elementary Practical Mathematics: "Academic methods of teaching mathematics succeed with about five per cent of all students, the small minority who are fond of abstract reasoning; they fail altogether with the average student. Mathematical study may be made of great value to the average man if only it is made interesting to him." Here is the real reason for the pressure there is upon us. We deal in the abstract and in the rigorous; the average person does not, and to a certain extent cannot. We teach the wrong way,—let us quote again: "There is always a difficulty in obtaining competent teachers (of practical mathematics). Any man who has learnt pure mathematics is thought by himself and others to be fit to teach, whereas his very fondness for and his fitness to study pure mathematics make it difficult for him to understand the simple principles underlying the new method. The average boy cannot take to abstract reasoning, and he is called stupid; I think him much wiser than the boy who is usually called clever."

We may not believe any of this stuff, we may force it out of our consideration; but there are many who believe it all, and they will constantly bring it back to our attention. And we cannot compromise more than temporarily by abridging our course; the very abridgment will produce less efficiency in the sort of thing we do teach. Even an average class will take great delight in hard differentiations and integrations, they will rejoice in conquering the difficulty, as I many times observed in the classes of A. W. Phillips at Yale,—provided the class is drilled in differentiation and integration until the majority acquire sufficient technique to make the game interesting. It is ability to do that maintains the interest. When we abridge our course without otherwise changing it we diminish the chances that the student shall become able to do what we teach him. That is the weakness of mere abridging. Diminishing the requirement in Greek for entrance to college killed preparatory Greek as quickly as anything could.

For ourselves, we do not believe in going the whole way with Perry; we believe that some abstract reasoning is good, and with our students prepared as they are when they come to us from the secondary schools a certain amount of abstract reasoning is not only good but possible. If we can follow a short course in calculus from a book like Snyder and Hutchinson's by a considerable course in concrete and practical problems, that may be our best procedure. But if we are to be allowed altogether only a short course, we should make that much less mathematical in the canonical sense; and by doing so we may perhaps be entrusted with a greater allotment of time.

Davis's Calculus is a frank attempt to introduce variety and interest into the calculus. The work therefore has attractive elements; one may easily exclaim: How inspiring to teacher and pupil to have all this constant contact with nature! That the book has bad qualities is obvious to anybody examining it carefully, but it is only after the sad disillusionment of teaching it a year that one can really find out how largely the bad outweighs the good. The book will therefore have many enthusiastic adopters and many speedy rejectors.

The main difficulty is that careless workmanship (or playmanship) permeates the whole in such an insidious fashion that it is partly hidden to the prospective user and always a burden to the actual user. Whether author, assistant author, or editor is responsible for this defect we cannot say; but it is improbable that any real hard cooperation by all upon the whole could have left so many lesions, and we may guess that one brewed the text, another peppered in the exercises, and a third sprinkled in the sage advice to Dear Reader and the gratuitous reflections. The answer book is full of errors, and thus is a great annoyance to the serious student, a corrupter of the careless worker. A table is valueless except as it is accurate, yet inaccuracies are found in the formula for center of pressure (not given in the text) and in the polar equation of the cissoid.

If Davis-Brenke-Hedrick had written a sufficiently original text we could pardon a number of errors, even under triple responsibility; but there is no particular originality about the work. They treat the algebraic function first, both as regards integration and differentiation, and when they come to transcendental functions they carry on the differential and integral calculus simultaneously. But so did Mercer in 1910; and if we may trust a comparative judgment of two books one of which we have not taught, we should unhesitatingly say that Mercer, though bearing but one workman's name, is incomparably the more careful and valuable production. And, to mention no others, Byerly in his Differential Calculus as long ago as 1879 introduced the integral calculus early and carried it along with the differential. It can hardly be expected that Byerly's book as it stands after 35 years should appeal strongly to teachers of the present day; yet its plan has many of the good features of recent books which try to freshen up the calculus.

The authors include a considerable treatment of differential equations as did Mercer in 1910 and Woods and Bailey in their Course in Mathematics, volume 2, some years before. This is not a bad thing to do, for the integration of the simpler differential equations is a natural part of the integral calculus; but I am very strongly of the opinion that a systematic chapter on rectilinear dynamics, such as is found in Byerly's and Osgood's books, is of much more value to the student and teaches him just as much about differential equations. One of the earliest differential equations we meet is the so-called compound interest law, for which our authors use the term snowball law as more " suggestive "; it is too bad that they regard it as more suggestive, for it is very wide of the mark. (The snowball law makes an interesting exercise for engineers; it leads under reasonable assumptions, such as an engineer would make, to an algebraic integral.)

One thing above all others students of mathematics should learn, namely, that it is the business of mathematics to teach them to think and talk coherently. They should have clean-cut definitions and straightforward proofs. If a book contains these, the teacher may safely be trusted to furnish the discussion and to point out the beauties of the landscape; whereas it is difficult for the teacher to furnish the definition and proof with any effectiveness to a class whose text is mostly discussion. The authors' book is long despite its moderate number of pages; for it is printed to a large extent in eight-point type. It could be abridged without substantive omissions.

It is the business of sets of exercises to offer definite and somewhat graded problems to the student; but the exercises in this text are wholly ungraded and in many cases very indefinite in statement. Sometimes there is scarcely a single exercise in a whole set which I should wish to assign to a class, and in every case it is necessary to take the greatest care in the selection of exercises. We recommend a careful reading of the lists of exercises to every prospective user.

The practice of leaving important items for exercises, a plan of which I heartily approve in advanced texts where exercises are hard to find or where we are trying to teach the student research, seems very doubtful for any elementary text. Yet here the area of a surface of revolution, component accelerations, hyperbolic functions, average values, centers of gravity, the element of volume in polar or spherical coordinates are thus left as exercises. It must be a remarkable class that can work out for itself the polar element of volume; it is more than most of us can do to draw a decent figure and give the proof for the class.

We have heard a great deal off and on about the necessity of giving the student power and the spirit of investigation; but this is merely a visionary's ideal, as anybody can see by pondering upon the question: How many of our doctors of philosophy in mathematics in this country or abroad are engaged in real research? If six to eight years of training lapse into desuetude in the case of professional mathematicians, what can you expect to accomplish with sophomores? Put into the text what you want them to know in such form that they can learn it, say I, and then see that they do learn it. And I have heard a very eminent investigator recommend the same sort of thing for candidates for the doctorate.

There is no need of going into the details, whether bad or good, of Davis's text. Suffice it to say that if books are not more carefully written, we shall have to refrain from adopting them from very lack of time to examine them in sufficient detail to make it safe to adopt them; but it is too bad to throw the whole reponsibility upon the user instead of upon the author and publisher, where we previously thought it belonged, at least to a very large extent.

Vivanti's book of exercises, a companion to his Lezioni d'analisi infinitesimale, contains 575 well-selected solved exercises in calculus; there are no applications and no rigorous types. The list should be of value to American writers of texts, but it is difficult to see how it can be of direct use in our classes.

EDWIN BIDWELL WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

NOTES.

THE April number (volume 16, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "Quartic curves modulo 2," by L. E. DICKSON; "Mixed linear integral equations of the first order," by W. A. HURWITZ; "Prime power groups in which every commutator of prime order is invariant," by W. B. FITE; "On the order of primitive groups," by W. A. MANNING; "A proof of the invariance of certain constants of analysis situs," by J. W. ALEXANDER, II; "Point sets and allied Cremona groups," by A. B. COBLE; "Scroll directrix curves," by C. T. SULLIVAN.

THE April number (volume 37, number 2) of the *American Journal of Mathematics* contains: "Invariantive theory of plane cubic curves modulo 2," by L. E. DICKSON; "On the projective differential classification of n-dimensional spreads generated by ∞^1 flats," by ARTHUR RANUM; "On the order of a restricted system of equations," by F. F. DECKER; "One-parameter families of curves," by L. P. EISENHART; "On a porism connected with the theory of Maxwell's equations and a method of obtaining the lines of electric force due to a moving point charge," by H. BATEMAN; "The abstract definitions of groups of degree 8," by JOSEPHINE E. BURNS.

THE Association of mathematics teachers of New Jersey held its second regular meeting at Trenton on May 1. Papers were read by OSWALD VEBLEN: "The affine geometry"; RICHARD MORRIS: "The auxiliary angle"; FLETCHER DURELL: "Mathematics and efficiency"; A. W. BELCHER: "First year mathematics for a technical high school"; C. O. GUNTHER: "Trigonometry for the college student"; J. W. COLLITON: "The study conference plan in mathematics"; H. E. WEBB: "Geometric definition of the trigonometric functions" and "Outline of a course in advanced commercial algebra." Professor H. B. FINE was elected president of the association.

THE Adams prize for the year 1913–1914 has been awarded to G. I. TAYLOR (Trinity, 1910). The title of the prize memoir is: "The phenomena of the disturbed motion of fluids, including the resistances encountered by bodies moving through them." The value of the prize is about £250.

THE early publication of an Analytic Mechanics, by J. A. MILLER and S. B. LIBBY, of Swarthmore College, is announced by D. C. Heath and Company.

UNIVERSITY OF CALIFORNIA.—The following advanced courses are announced for the summer session, June 21 to July 31:—By Professor M. W. HASKELL: Constructions by ruler and compass.—By Professor C. J. Keyser: Analytical geometry of three dimensions; History and significance of central mathematical ideas and doctrines.—By Professor H. W. Tyler: History of mathematical science.—By Professor T. M. Putnam: Three famous problems of geometry; Theory of functions of a complex variable.

Dr. E. S. Allen, of Brown University, has been appointed instructor in mathematics at the University of Michigan.

Professor J. J. Hardy, who had held the chair of mathematics and astronomy at Lafayette College since 1876, died May 2, 1915, at the age of seventy-one years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Cajori (F.). History of mathematics. 2d reprint. New York, Macmillan, 1913. 8vo. 422 pp. $3.50

Campbell (D. F.). A short course on differential equations. 5th reprint. New York, Macmillan, 1913. 12mo. 123 pp. $0.90

——. Elements of the differential and integral calculus. 4th reprint. New York, Macmillan, 1913. 12mo. 362 pp. $1.90.

Carvallo (E.). Le calcul des probabilités et ses applications Paris, Gauthier-Villars, 1912. 8vo. 9+169 pp.

Cremona (L.). Opere matematiche. Tomo II. Milano, Hoepli, 1915. 4to. 459 pp. L. 25.00

Encyklopädie der mathematischen Wissenschaften. Band II 2, Heft 4: E. Hill, Lineare Differentialgleichungen im komplexen Gebiet. Leipzig, Teubner, 1915. Gr. 8vo. Pp. 471–562. M. 2.80

Fine (H. B.) and Thompson (H. D.). Coordinate geometry. 5th reprint. New York, Macmillan, 1914. 12mo. 300 pp. Half leather. $1.60

Haeussler (J. W.). Geometrischer und algebraischer Beweis des Fermatschen Satzes durch Ausziehen der n-ten Wurzel und durch Ableitung des Satzes als ganzzahlige Ungleichung. Berlin, M. Krayn, 1912. 8vo. 48 pp.

Hill (E.). See Encyklopädie.

La Marca (R.). Criterî di congruenza e criterî di divisibilità. Esercizî varî. Torre del Greco, Pantaleo, 1912. 8vo. 30 pp.

Laska (W.). Einleitung in die geometrische Funktionentheorie. 2te Auflage. Bremerhaven, 1915. Gr. 8vo. 6+87 pp. M. 2.00

Nekrassow (P. A.). Theory of probabilities. 2d edition, with the addition of the statistical theory of correlation and the elements of nomography. (In Russian.) St. Petersburg, 1912. 8vo. 36+532 pp.

Riggs (N. C.). Analytic geometry. 8th reprint. New York, Macmillan, 1914. 12mo. $1.60

Thompson (H. D.). See Fine (H. B.).

WIESNER (S.). Ueber mathematische Logik. Berlin, 1914. Gr. 8vo. 12 pp.

II. ELEMENTARY MATHEMATICS.

BEARD (W. S.). Arithmetical examples. London, Dent, 1912. 269 pp.

COMFORT (L. C.). Geometry: a reasoned chain. London, Gill, 1915. 1s.

CONTI (A.). Elementi di calcolo letterale. 3a edizione. Bologna, Zanichelli, 1912. 16mo. 136 pp.

HALL (H. S.) and STEVENS (F. H.). Examples in arithmetic taken from "A school arithmetic." London, Macmillan, 1912. 8vo. 15+281+39 pp.

HOČEVAR (F.). Lehr- und Uebungsbuch der Arithmetik für Gymnasien und Realgymnasien. Oberstufe. 2te umgearbeitete Auflage. Wien, F. Tempsky, 1912. 152 pp.

JENSEN (O.). See WALTHER (F.).

LESSER (O.). See SCHWAB (R.).

NASAREVSKY (J.). Die Anzahl der Ziffern in der Periode des periodischen Dezimalbruches (Vortrag gehalten in der Sitzung der Mathematischen Gesellschaft Charkow 15/28. IV. 1909). Charkow, 1912. 8vo. 6 pp.

OTTO (F.), PETRI (W.), THAER (A.) und ZIEGLER (J.). Mathematik fur Oberlyzeen. In 2 Teilen. 1ter Teil unter Mitwirkung von Otto, bearbeitet von Petri, Thaer, Ziegler. 2te umgearbeitete Auflage. Leipzig, F. Hirt, 1912. 8vo. 7+293 pp.

PETRI (W.). See OTTO (F.).

ROTHROCK (D. A.). Elements of plane and spherical trigonometry, with tables. 3d reprint. New York, Macmillan, 1912. 8vo. 147+99 pp. $1.40

ROUWOLF (R.) See THAER (A.).

SCHWAB (R.) und LESSER (O.). Mathematisches Unterrichtswerk zum Gebrauche an hoheren Lehranstalten. 2te Auflage. Leipzig, Freytag, 1912. 8vo. 290 pp.

SOPOTSKO (L.). The accuracy of computation with logarithmic tables. (In Russian.) Moscow, 1912. 8vo. 12pp.

SPORER (B.). Niedere Analysis. 2te Auflage. 4ter Abdruck. Berlin und Leipzig, Goschen, 1912. 8vo. 179 pp.

STEVENS (F. H.). See HALL (H. S.).

THAER (A.). See OTTO (F.).

THAER (A.) und ROUWOLF (R.). Rechenbuch fur höhere Schulen. Ausgabe B, in drei Heften und einem Ergänzungsheft. 1tes Heft, für Sexta, 3te und 4te Auflage, 1913; 2tes Heft, für Quinta, 3te und 4te Auflage, 1914; 3tes Heft, fur Quarta und Untertertia, 3te und 4te Auflage, 1914; Erganzungsheft, für Obertertia und Untersekunda, 2te Auflage, 1913. Breslau, F. Hirt. 95+96+96+100 pp. Kartoniert. M. 1.00+1.00+1.00+1.00

——. Rechenbuch fur Vorschulen hoherer Lehranstalten in drei Heften. 3te Auflage. Breslau, F. Hirt, 1914. 83+95+100 pp. Kartoniert. M. 0.80+0.90+1.00. Ergebnisse zum 3ten Heft. M. 0.60

THAER (A.) und WIMMENAUER (T.). Arithmetische Aufgaben fur hohere Schulen. Ausgabe C, für Realschulen. Breslau, Hirt, 1912. 8+160 pp. Geb. M. 2.40. Ergebnisheft dazu. Geh. M. 1.60.

TOROPOV (K.). Magic series and their application to the solution of exercises. (In Russian.) 2d edition. Orenburg, 1912. 8vo. 45+2 pp.

WALTHER (F.) und JENSEN (O.). Mathematischer Lehr- und Uebungsgang fur hohere Mädchenschulen, Lyzeen und Studienanstalten. Abtl. A 3 (Kl. VII–V). 3tes Heft (Kl. V) von Jensen. Leipzig, F. Brandstetter, 1912. 8vo. 140 pp.

WIMMENAUER (T.). See THAER (A.).

ZIEGLER (J.). See OTTO (F.).

III. APPLIED MATHEMATICS.

BIÉMONT (P.). See CÉSAR (J.).

CALDARERA (F.). Corso di meccanica razionale. Vol. I: preliminari, cinematica, studio delle forze. 2a edizione migliorata. Palermo, Virzi, 1915. 8vo. 8+336 pp. L. 13.00

CASTLE (F.). Workshop arithmetic. London, Macmillan, 1915. 1s. 6d.

CÉSAR (J.) et BIÉMONT (P.). Traité d'arithmétique financière. Paris, Hermann, 1912. 8vo. 170+36 pp.

ENCYKLOPADIE der mathematischen Wissenschaften. Band IV 2 II, Heft 5: K. Wieghardt, Theorie der Baukonstruktionen, II; E. Hellinger, Allgemeine Ansatze der Mechanik der Kontinua; T. Karman und L. Foppl, Physikalische Grundlagen der Festigkeitslehre. Leipzig, Teubner, 1914. Gr. 8vo. Pp. 15+535–770. M. 8.00

FESTSCHRIFT W. Nernst zu seinem 25 jährigen Doktorjubiläum gewidmet von seinen Schulern. Halle, W. Knapp, 1912. Gr. 8vo. 6+487 pp.

FOCK (E.). See KLESSNER (E.).

FÖPPL (L.). See ENCYKLOPADIE.

HELLINGER (E.). See ENCYKLOPÄDIE.

IWERONOV (I.). Die Methode der kleinsten Quadrate. Vorlesungen gehalten im Konstantinischen Feldmessinstitut und Moskauer Agronomischen Institut. 2te Auflage. Moscow, 1912. 8vo. 136 pp.

KARMAN (T.). See ENCYKLOPÄDIE.

KLESSNER (E.) und FOCK (E.). Physik und Chemie für Lyzeen und hohere Mädchenschulen sowie fur den Anfangs-Unterricht in Studienanstalten. Heft 1: Mechanische Grundbegriffe; Mechanik starrer Körper, 1ter Teil; Wärmelehre. Berlin, Salle, 1914. 71 pp. Kartoniert. M. 1.50

MANSION (P.). L'avantage du banquier au jeu de baccara. Louvain, Ceuterick, 1912. 12mo. 8 pp.

NERNST (W.). See FESTSCHRIFT.

THIRY (C.). Etude pratique et usuelle des emprunts, obligations. Gand, Van Goethem, 1912. 8vo. 76 pp.

VAUCHER (A.). Notes additionelles à la théorie de l'échelle musicale. Paris, Gauthier-Villars, 1914. 8vo. 16 pp. Fr. 0.75

WIEGHARDT (K.). See ENCYKLOPÄDIE.

ZAUBRECHER (F. X.). Die Lebensversicherung. (Sammlung Kosel, Band 53.) Kempten, J. Kösel, 1912. Sm. 8vo. 12+202 pp.

THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY IN NEW YORK.

The one hundred and seventy-seventh regular meeting of the Society was held in New York City on Saturday, April 24, 1915, the following seventy-one members being in attendance at the two sessions:

Mr. J. W. Alexander, II, Professor M. J. Babb, Dr. Ida Barney, Dr. F. W. Beal, Dr. R. D. Beetle, Professor Susan R. Benedict, Mr. A. A. Bennett, Professor E. G. Bill, Professor G. D. Birkhoff, Professor Maxime Bôcher, Professor Joseph Bowden, Professor E. W. Brown, Dr. T. H. Brown, Professor A. B. Coble, Dr. Emily Coddington, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Professor Elizabeth B. Cowley, Dr. Louise D. Cummings, Dr. H. B. Curtis, Mrs. E. B. Davis, Professor L. P. Eisenhart, Professor H. B. Evans, Professor F. C. Ferry, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor A. S. Gale, Professor O. E. Glenn, Dr. G. M. Green, Professor C. C. Grove, Professor H. E. Hawkes, Professor E. V. Huntington, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Dr. L. M. Kells, Professor C. J. Keyser, Dr. E. A. T. Kircher, Professor W. R. Longley, Professor James Maclay, Dr. E. J. Miles, Professor H. H. Mitchell, Professor E. H. Moore, Dr. R. L. Moore, Mr. G. W. Mullins, Professor W. F. Osgood, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor A. D. Pitcher, Dr. H. W. Reddick, Professor R. G. D. Richardson, Dr. R. B. Robbins, Professor W. H. Roever, Professor J. E. Rowe, Dr. Caroline E. Seely, Professor H. E. Slaught, Professor Clara E. Smith, Professor H. F. Stecker, Professor H. D. Thompson, Mr. H. S. Vandiver, Mr. C. E. Van Orstrand, Professor Oswald Veblen, Mr. R. A. Wetzel, Professor H. S. White, Miss E. C. Williams, Professor Ruth G. Wood, Professor J. W. Young.

Owing to the large amount of business to be transacted, the Council met on the evening of April 23, as well as on the following morning. President E. W. Brown occupied the chair at the Council meeting, being relieved at the sessions of the Society by Ex-Presidents Osgood, Moore, and White, and Vice-President Veblen. The Council announced the election of the following persons to membership in the Society: Professor L. S. Hill, University of Montana; Miss G. I. McCain, Indiana University; Mr. J. F. Riley, Rice Institute. Eleven applications for membership in the Society were received. Professor P. F. Smith was reelected a member of the Editorial Committee of the *Transactions*, to serve for the three years beginning October 1, 1915.

The invitation of the Division of Mathematics of Harvard University to hold the summer meeting and colloquium of the Society at Harvard University in 1916 was accepted. A committee, consisting of Professors Osgood, Bôcher, Moore, White and the Secretary, was appointed to make the arrangements for this meeting. A committee was also appointed to prepare a list of nominations for officers and other members of the Council to be elected at the annual meeting. It was decided to hold the annual meeting this year in New York at such time as not to interfere with the meeting of the Chicago Section at Columbus in affiliation with the American association for the advancement of science.

Reports of committees were received concerning the proper attitude of the Society toward the movement against mathematics in the schools and the possible relations of the Society to the field now covered by the *American Mathematical Monthly*. In the former matter it was held to be inadvisable for the Society to take any action at the present time. Regarding the relations of the Society to the *Monthly*, the sense of the Council was embodied in the following resolution:

"It is deemed unwise for the American Mathematical Society to enter into the activities of the special field now covered by the *American Mathematical Monthly;* but the Council desires to express its realization of the importance of the work in this field and its value to mathematical science, and to say that should an organization be formed to deal specifically with this work, the Society would entertain toward such an organization only feelings of hearty good will and encouragement."

In the interval between the sessions the members took luncheon together. Thirty-seven members gathered in the evening for an enjoyable dinner at the Terrace Garden.

The following papers were read at this meeting:

(1) Professor C. J. DE LA VALLÉE POUSSIN: "Démonstration simplifiée d'un théorème de Vitali sur le passage à la limite sous le signe d'intégration."

(2) Dr. C. A. FISCHER: "Minima of double integrals with respect to one-sided variations."

(3) Dr. G. M. GREEN: "Nets of space curves."

(4) Dr. R. L. MOORE: "A set of axioms in terms of point, region, and motion."

(5) Dr. R. L. MOORE: "On the categoricity of a set of postulates."

(6) Mr. H. S. VANDIVER: "A property of cyclotomic integers and its relation to Fermat's last theorem. Second paper."

(7) Mr. J. F. RITT: "The reduction of invariant equations."

(8) Professor E. B. WILSON: "Linear momentum, kinetic energy, and angular momentum."

(9) Professor F. H. SAFFORD: "An irrational transformation of the Weierstrass $\wp$-function curves."

(10) Professor ARNOLD EMCH: "A certain class of functions connected with Fuchsian groups."

(11) Professor G. D. BIRKHOFF: "An elementary double inequality for the roots of an algebraic equation having greatest absolute value."

(12) Professor H. S. WHITE and Dr. LOUISE D. CUMMINGS: "Groupless triad systems on 15 elements" (preliminary report).

(13) Professor EDWARD KASNER: "Conformal geometry in the complex domain."

(14) Professor EDWARD KASNER: "Convergence proofs connected with equilong invariants."

(15) Professor E. V. HUNTINGTON: "Notes on the catenary, including the case of an extensible chain."

(16) Mr. R. C. STRACHAN: "Note on the catenary."

(17) Professor J. E. ROWE: "A property of the osculant conic of the rational cubic."

(18) Professor J. E. ROWE: "The node of the rational cubic as a rational curve in lines."

(19) Dr. F. W. BEAL: "A congruence of circles."

(20) Professor H. F. STECKER: "Linear integral equations whose solutions have only a finite number of terms."

(21) Mr. C. A. EPPERSON: "Note on Green's theorem."

(22) Mr. L. J. REED: "Some fundamental systems of formal modular invariants and covariants."

(23) Mr. J. R. Kline: "Double elliptic geometry in terms of point and order."

(24) Dr. Alexander Pell: "On the curves of constant torsion."

(25) Mr. J. H. Weaver: "The Collection of Pappus" (preliminary report).

(26) Professor J. W. Young and Dr. F. M. Morgan: "The geometries associated with a certain system of Cremona groups."

(27) Professor T. H. Gronwall: "A functional equation in the kinetic theory of gases."

(28) Mr. J. W. Alexander, II: "A theorem on conformal representation."

Professor de la Vallée Poussin was introduced by Professor Osgood, Mr. Ritt by Professor Kasner, Mr. Strachan by Professor Huntington, Mr. Reed by Professor Glenn, Mr. Kline by Dr. R. L. Moore, and Mr. Weaver by Professor Babb. Mr. Epperson's paper was communicated by Professor G. C. Evans. In the absence of the authors the papers of Professor Wilson, Professor Safford, Professor Emch, Mr. Epperson, Professor Young and Dr. Morgan, Professor Gronwall, and Mr. Alexander were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor de la Vallée Poussin presented a new proof of a theorem due to Vitali. This proof forms part of a paper entitled "Les intégrales de Lebesgue" which will appear in the *Transactions* of the Society at a later date.

2. In some problems in minima of double integrals, the surface over which the integral is taken is restricted to lie in a given closed region R. Then it may happen that there is no extremal surface bounded by a previously given space curve which lies entirely within R, but that there is a surface consisting of a part of an extremal surface and a part of the boundary of R which minimizes the given integral. In this paper Dr. Fischer derives some necessary conditions for such a minimum, and also a set of sufficient conditions.

3. In the metric differential geometry of a surface, use is frequently made of the Gaussian parameter form of representa-

tion, the surface being thereby referred to an arbitrary net of parameter curves. But only the organic properties of the surface, and not those of the net, are the objects of study. In this paper Dr. Green investigates an arbitrary net of curves for its own sake. A canonical development for the configuration in the neighborhood of a point is found, and leads to a consideration of certain osculating quadric and cubic surfaces. Of especial interest and importance is a certain congruence, formed by the lines of intersection of the osculating planes to the curves of the two families of the net at the points of the surface. The same congruence was used in obtaining the canonical development of a surface referred to a conjugate net, given in the report of the February, 1914, meeting of the Society. The study of this congruence leads to certain theorems concerning the general theory of congruences, in which the focal points on a line of the congruence appear as the double points of an involution. New theorems of a metric nature, concerning geodesics and normal congruences, are corollaries of the general theory. The entire discussion is founded on the consideration of a completely integrable system of partial differential equations, following out the general ideas of Wilczynski.

4. Dr. Moore proposes the following set of axioms for plane analysis situs in terms of point and region.

1. A region is a set of points.

2. Every region contains at least two points.

Definitions. P is a limit point of M if every region which contains P contains at least one point of M different from P. A set of points is connected if, however it be divided into two subsets, one of them contains a limit point of the other one. The boundary of a region R is the set of all those limit points of R that do not belong to R. If R is a region, R' denotes R plus its boundary.

3. There exists a countably infinite set of regions R_1, R_2, $R_3 \cdots$ such that (1) if P and X are distinct points within a region R, then there exists n_{PR} such that if $n > n_{PR}$ and R_n contains P then R_n' is a proper subset of $R - X$; (2) if $\delta > 0$ and P is a point, then there exists n greater than δ such that R_n contains P.

4. Each region is connected and so is its exterior.

5. If two regions contain a point in common, then they

also contain in common a region which contains that point.

6. If the point P does not belong to the region R, then every region containing P contains a region which has no point in common with R.

7. Not every infinite set of points has a limit point.

8. If A and B are distinct points on the boundary of a region R and $A \times B$ is a Jordan arc whose interior t lies entirely within T (where T is either R or its exterior) and $T - t = S_1 + S_2 + S_3$, where S_1, S_2, S_3 are mutually exclusive point sets, then one of the sets S_1, S_2, S_3 contains a limit point of one of the other two.

9. The Heine-Borel proposition as applied to a set of regions covering a given region and its boundary.

10. If R is a region, T either its interior or its exterior, B a point in T or on the boundary of R, A a point in T, and $A \times B$ a Jordan arc whose interior t lies in T, then if T is connected so is $T - t$.

11. Every Jordan arc is the boundary of only one region.

12. If one point of a region R is interior to a Jordan curve lying in R, then every point of R is interior to such a curve.

From axioms 1–12 follows a considerable body of propositions relating to non-metrical properties of plane curves and point sets. It follows from axioms 1–6 and 9 that every two points of a region can be joined by a Jordan arc lying in that region. If axiom 12 is omitted and in place of axiom 3 are substituted certain axioms involving the third undefined term motion, there results a system of axioms categorical with respect to point, motion, and limit point of a point set as defined above.

5. In a recent paper* Dr. Moore has given a system of axioms for the linear continuum in terms of point and limit. In his second paper Dr. Moore proposes to show that, while this set of axioms is categorical with respect to point and order as there defined, nevertheless it is not categorical with respect to point and limit but will become so if for axioms 5 and 8 are substituted the following axioms 5′ and 8′:

5′. If r_1 and r_2 are two mutually exclusive rays and if the set S of all points contains points not belonging to $r_1 + r_2$, then every infinite set of points lying in $S - (r_1 + r_2)$ has at least one limit point.

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* "The linear continuum in terms of point and limit," *Annals of Mathematics*, vol. 16 (March, 1915), pp. 123–133.

8′. If P is a limit point of M, then P is a limit point of every infinite subset of M.

6. In a paper presented at the January meeting of the Society, Mr. Vandiver proved a relation in connection with Fermat's last theorem from which the criteria of Kummer and Furtwängler were deduced. The present paper contains developments along the same line.

7. In computing invariants under a group by executing a transformation of the group and then eliminating the parameters of the transformation, one obtains a system of equations which, when properly separated, yield the invariants. It seems that the only proof heretofore known of the separability of the invariant equations consists in an appeal to the results of the Lie theory. Mr. Ritt gives a direct method for replacing the invariant equations by a system of invariants.

8. Professor Wilson points out that some persons, who should know better, seem to think that the *kinematic* resolution of motion of laminas and rigid bodies into a velocity of translation of any arbitrary point and an angular velocity about an axis through that point, is *dynamically* valid. As a matter of fact the center of mass is introduced in kinetics precisely so that we may have a point for which the kinematic resolution holds dynamically. A number of simple geometric theorems develop from the search after points other than the center of mass for which dynamic results have the same simple resolution as the kinematic. The paper will appear in the *American Mathematical Monthly.*

9. On September 10, 1912, Professor Safford read before the Society a preliminary communication entitled "An irrational transformation of the Weierstrass $\wp$-function curves." The present paper gives the final results and will be sent to the *Archiv der Mathematik und Physik,* in which the first paper appeared on June 12, 1914. The original curves are obtainable from $x + iy = \wp(t + iu)$, and are of the fourth degree. After transformation and rationalization the resulting equation is of the sixteenth degree, but is resolvable into three factors,

corresponding to one irresolvable curve of the eighth degree and one of the fourth degree taken twice.

10. In this paper Professor Emch shows that the series

$$\sum_{\lambda=0}^{\infty}[R\{(z_\lambda - a_\lambda)(z_\lambda - b_\lambda)\} - R\{(z_\lambda' - a_\lambda')(z_\lambda' - b_\lambda')\}],$$

extended over a Fuchsian group with the unit circle as a natural boundary, where z, z', a, b, a', b' are within and not on the unit circle, z', a', b' are fixed, and $R(z)$ is a rational function, under certain restrictions represents an analytic function of z with the unit circle as a natural boundary. It appears that, in general, these functions are not automorphic in the ordinary sense of the word.

11. The note by Professor Birkhoff appears in the present number of the BULLETIN.

12. From previous publications and a paper presented to this Society in October, 1914, 44 different triad systems on 15 elements are known. These 44 belong to groups of order from $8!/2$ down to 2. Systems whose group is identity, or groupless systems, on 15 elements have not been known hitherto. This paper of Professor White and Dr. Cummings exhibits a mode of deriving new systems from old, and applies it to the derivation of groupless systems. Over 20 such are announced, and the extent of the research now under way is indicated.

13. Real conformal geometry deals with the ∞^2 real points of the usual Gaussian plane under real conformal transformation. Complex conformal geometry, as discussed by Professor Kasner, deals with the totality of ∞^4 complex points of the plane under the larger group of complex conformal transformations. In earlier papers (see Proceedings of the International Congress of Mathematicians, Cambridge, volume 2, pages 81–87) it was shown that in the real plane a regular analytic curvilinear angle has an invariant of higher order only when its magnitude is a rational part of 360° (including the horn angles of magnitude 0). The present paper includes a complete classification of analytic angles in the complex plane. In contrast with the real discussion, it is shown that certain

types of angle have an infinite number of higher invariants, and other types are capable of uniquely determining a conformal transformation. The bisection problem (defined in the paper cited) is also generalized and it is shown that the number of solutions may be not only 0, 1, 2, ∞, as in the real case, but also 3, 4, or any finite number.

14. In the paper cited above, Professor Kasner discussed also the invariants of a pair of analytic curves having a common tangent under the group of equilong transformations of the plane. It was shown that each type of configuration (classified according to the order of contact of the two curves) has a unique invariant J_{2k+1}. Hence if two configurations of the same type have the same J it follows that a transformation converting the one into the other can be formally calculated as a power series. The author now completes the theory by showing that these series are always convergent. This is done by reducing the problem to a differential equation of the form $dy/dx = P/Q$, where P and Q are series in x and y without absolute terms. The regular Cauchy theory is thus not applicable, but it is shown that after certain substitutions the Briot-Bouquet criterion may be used. The equilong theory thus turns out to be essentially simpler than the (roughly) dual conformal theory, since in the latter case the convergence question cannot be reduced to differential equations.

15. Professor Huntington's paper shows how to compute the length of a chord of a generalized catenary from the following data: the inclination of the chord to the horizontal; the weight per unit length, area of cross-section, and modulus of elasticity of the tape; the nominal length of the arc; and either the tension at the upper end or the tension at the lower end; and tables are provided for facilitating the computation. The paper also contains tables giving an auxiliary variable z in terms of the arguments $(\cosh z - 1)/z$ or $z/\cosh z$ or $\sinh z/z$, these tables being useful in the ordinary computations connected with the common catenary.

16. Mr. Strachan points out the desirability of having tables for the quantities $l = u/\sinh u$ and $k = 2u/(\cosh u - 1)$ for use in certain well-known computations connected with the catenary, and suggests the following form for such a table:

$\theta = gd\,u$	u	$\sinh u$	$\cosh u$	$\tanh u$	l	k

where the argument θ runs from 0° to 90° at intervals of 20″, and the functions are tabulated to six figures.

As an alternative, he suggests the following form, to be used in connection with existing trigonometrical tables:

$\theta = gd\,u$	$\log u$	$\log l$	$\log k$

17. In Professor Rowe's first paper the osculant conic of the rational cubic is defined algebraically, and it is proved not only that this osculant conic at a particular point whose parameter is t_1 touches the R^3 at that point (which is a well-known theorem), but that the parameters of the remaining intersections of the R^3 and the osculant conic constitute a self-apolar set.

18. In this paper Professor Rowe shows how to derive from the parametric equations of the rational cubic the equation of the node in line coordinates. The parametric equations of the node in lines may be found by taking the polar of the flex cubic as to the line equation of the curve.

19. Dr. Beal considers a congruence of circles which he calls a congruence C and which may be defined in the following manner: Any point P on a circle of the two-parameter system is to be a point P of a surface Σ whose tangent plane at P passes through the center of the circle and cuts the plane of the circle under an angle which is a function of the circle alone. Of particular interest is a congruence C such that the center of the circle is the point at which the plane of the circle touches its envelope M. In this case the surfaces Σ have the same total curvature as surface M. The surfaces Σ are not such that they admit a congruence C_1 such that M is one of an infinity of surfaces related to C_1 in the same manner as Σ is to C, unless M is pseudospherical. Then the surfaces Σ are of course Bäcklund transforms of M.

20. In solving actual problems of linear integral equations the Fredholm series $D(s, t, \lambda)$ and $D(\lambda)$ often have only a finite number of terms. It is the purpose of Professor Stecker's paper to determine the conditions under which this takes place. It is shown that it is necessary and sufficient

that the kernel be of the form $\sum_{i=1}^{i=n} \varphi_i(s)\psi_i(t)$, except, possibly, at the points of a set of content zero; or that the known term in the integral equation itself satisfy a certain integral equation. The corresponding forms of solution of this linear integral equation are also obtained.

21. The aim of Mr. Epperson's paper is to extend Green's theorem to apply to integro-differential relations equivalent to the general linear partial differential equation of the second order in two variables. These relations are written in such a way as not to involve derivatives of the second order, and the theorem is proved without assuming their existence. This point of view is desirable for the possibility of its application to physics. The proof is deduced without making use of Green's theorem in its usual form.

22. The method of constructing formal modular concomitants of a form that Mr. Reed uses in this paper is the one outlined by Professor O. E. Glenn in an article published in the BULLETIN for January, 1915. The method consists in finding transvectants of the form with the absolute covariants of the modular group and in the use of certain invariant operators. By this method are constructed fundamental systems for the linear form modulo p, for the quadratic form modulo 2, and a simultaneous system for the linear and the quadratic form. These three systems are shown to be complete. The same method applied to the quadratic form modulo p gives a series of irreducible concomitants, $\frac{1}{2}(p + 7)$ in number.

23. In his Rational Geometry, Halsted gives a set of axioms for double elliptic plane geometry in terms of point, order, association, and congruence. Mr. Kline proposes the following categorical set of eleven axioms for three dimensional double elliptic geometry in terms of point and order alone:

1. If A is a point, there exists at least one point A', different from A, such that the order $AA'C$ does not hold for any point C. A' is called an opposite of A.
2. There exist at least two distinct points A and B such that $B \neq A'$.
3. If ABC, CBA.

4. If ABC and A' is any opposite of A, $A'CB$.
5. If ABC and ACD, then not ADB.
6. If ABC and BCD, then either ABD or $AD'B$.
7. If there exist an infinite number of points, there exist three points A, B, D such that D is not on the line AB.
8. If AED and ABF and D is not on AB, then there exists an H such that BHD and EHF.

9 and 10 are the same as Axioms IX and X of Veblen's euclidean system.

11. The Dedekind cut postulate for every segment. The straight line AB is defined as the set of all points X satisfying one of the eight conditions $X = A$, $X = B$, $X = A'$, $X = B'$, XAB, AXB, ABX, $AX'B$.

It follows from Axioms 1–6 that a point has only one opposite; from Axioms 1 to 6 and Axiom 8 follows the existence of an infinite number of points and the fact that any two distinct points of a line, not opposites, determine this line. That the points of a line are in cyclical order follows from Axioms 1 to 6 and Axiom 8, Axiom 8 being merely used to show that, assuming the hypothesis of Axiom 5 satisfied, D is on the line AB. That every two lines in a plane intersect in two points, which are opposites, is a consequence of Axioms 1–8. It follows from Axioms 1 to 10 that every two planes intersect in a straight line.

24. Dr. Pell shows that the coordinates of a curve of constant torsion can be expressed as the sum of the coordinates of a minimal curve and of a spherical curve. The curve of constant torsion lies on a developable surface whose edge of regression is the corresponding minimal curve, the distance between corresponding points being constant. Some properties of a circular helix are generalized for curves of constant torsion.

25. Mr. Weaver has made a translation of the Collection of Pappus, to which he is adding a commentary, explaining in modern form the Greek geometric methods. In this preliminary paper he discusses the various criticisms of Pappus in the light of this extended study. He gives references from Pappus's own works, which have doubtless been missed by many historians who have blamed Pappus for literary theft or on the other hand have given credit which Pappus definitely disclaims. These ancient impressions being corrected, Mr.

Weaver wishes to discuss in succeeding papers some notions developed by Pappus which he believes are not generally known.

26. The paper by Professor Young and Dr. Morgan is a revision from a much more general point of view of a paper presented by them to the Society two years ago (April 26, 1913) under the title "The geometry associated with a certain group of Cremona transformations in space." This earlier paper called attention to a system $G_1, G_2, G_3, \cdots, G_n, \cdots$ of groups of Cremona transformations (of order n) in spaces of 1, 2, 3, $\cdots$, n dimensions respectively, and developed elementary properties of the geometry associated with one of the particular forms assumed by the group G_3 in space of three dimensions. In the present paper the symbol G (without subscripts) is used to denote any one of the groups G_i ($i = 1, 2, \cdots, n$); the associated point space, on which the transformations of G operate in a one-to-one reciprocal way, is denoted by S. The geometry of the system (S, G) is then developed. The theorems are for the most part the same as in the previous paper; the novelty in the present revision lies merely in the recognition (explicitly) of the fact that the theorems are of "general reference" in the infinite system of geometries defined by the groups G_i.

27. Let ξ, η, ζ, and ξ_1, η_1, ζ_1 be the velocity components of two spherical molecules at the instant of their collision, and $\varphi(\xi, \eta, \zeta)$ the logarithm of the function defining the distribution of velocities; then the Maxwell-Boltzmann fundamental theorem leads to the functional equation

$$\begin{aligned}&\varphi(\xi, \eta, \zeta) + \varphi(\xi_1, \eta_1, \zeta_1)\\ &\quad = f(\xi + \xi_1, \eta + \eta_1, \zeta + \zeta_1, \xi^2 + \eta^2 + \zeta^2 + \xi_1^2 + \eta_1^2 + \zeta_1^2).\end{aligned}$$

Assuming the existence of all six partial derivatives of the second order of φ, it is readily shown (Boltzmann, Vorlesungen über Gastheorie, volume 1, pages 128–131) that the general solution of the functional equation is

$$\varphi(\xi, \eta, \zeta) = a + b_1\xi + b_2\eta + b_3\zeta + c(\xi^2 + \eta^2 + \zeta^2)$$

with constant coefficients. In the present paper, Professor Gronwall establishes the same result under the sole assumption of the continuity of φ for all real values of the variables.

F. N. COLE,
Secretary.

AN ELEMENTARY DOUBLE INEQUALITY FOR THE ROOTS OF AN ALGEBRAIC EQUATION HAVING GREATEST ABSOLUTE VALUE.

BY PROFESSOR GEORGE D. BIRKHOFF.

(Read before the American Mathematical Society, April 24, 1915.)

LET there be given an algebraic equation of the nth degree written

$$x^n + C_{n,1}a_1x^{n-1} + C_{n,2}a_2x^{n-2} + \cdots + a_n = 0,$$

where $C_{n,1}$, $C_{n,2}$, $\cdots$ denote binomial coefficients. Let x_1, x_2, $\cdots$, x_n denote its roots, and X the greatest absolute value of any of these roots.

From the equations which express the coefficients in terms of the roots

$$C_{n,k}a_k = (-1)^k\Sigma x_1x_2\cdots x_k \quad (k = 1, 2, \cdots n)$$

we infer at once that

$$| a_k | \leqq X^k.$$

In fact there are precisely $C_{n,k}$ terms on the right-hand side of the kth equation, each of which is not greater than X^k.

Hence, if α stands for the greatest of the quantities

$$| a_1 |, \ | a_2 |^{1/2}, \ \cdots, \ | a_n |^{1/n},$$

it is clear that we have $X \geqq \alpha$.

Also from the given algebraic equation we obtain directly

$$| x |^n \leqq C_{n,1} | a_1 | | x |^{n-1} + \cdots + | a_n |,$$

where x represents any of the quantities x_1, x_2, $\cdots$, x_n. Replacing $| x |$ by one of its possible values X, and each quantity $| a_k |$ by the quantity α^k, at least as great, we obtain

$$| X |^n \leqq C_{n,1}\alpha | X |^{n-1} + \cdots + \alpha^n,$$

which may be written

$$X^n \leqq (X + \alpha)^n - X^n.$$

Transposing the term X^n to the left-hand side and extracting the nth root of the two members of the resulting inequality, we find

$$X \leqq \frac{\alpha}{2^{1/n} - 1}.$$

The roots of largest absolute value X are restricted by the double inequality

$$\alpha \leqq X \leqq \frac{\alpha}{2^{1/n} - 1},$$

where α denotes the largest of the quantities

$$| a_1 |, | a_2 |^{1/2}, \cdots, | a_n |^{1/n}.$$

The inequality $X \geqq \alpha$ was noted by R. D. Carmichael and T. E. Mason,* who observed also that the lower limit is reached if the equation is

$$(x + \alpha)^n = 0.$$

It is also evident that the upper limit found above is reached if the equation is

$$2x^n - (x + \alpha)^n = 0.$$

HARVARD UNIVERSITY,
April 23, 1915.

CERTAIN NON-ENUMERABLE SETS OF INFINITE PERMUTATIONS.

BY PROFESSOR A. B. FRIZELL.

(Read before the American Mathematical Society April 10 and December 28, 1914.)

1. THE simplest element of a permutation is the pairing of one of the objects permuted with a number indicating its place in the permutation. Such a pairing may be called a *primitive element* and denoted by (i, n), where n is the object and i the number of the place assigned to it. In this paper the objects will all be numbers, finite or transfinite.

2. Permutations of finite sets are simply collocations of primitive elements. They are conveniently denoted by ex-

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* BULLETIN, vol. 21 (1914), pp. 14–22; in particular p. 20.

pressions of the form $(i_1, n_1) \circ (i_2, n_2) \circ \cdots \circ (i_k, n_k)$ $(i_1 < i_2 < \cdots < i_k)$ and may be said to form *polynomial* elements. The number of the place will be called the *index* of a term (or monomial element) and the number placed there its *digit.* Index and digit may be transfinite and the terms may form an infinite series instead of a finite polynomial.

3. If in an infinite well ordered set only a finite number of objects are permuted, the pairing of these objects with their new places may be called a *finite permutation.* It is easy to see that any set of finite permutations can be well ordered. If, e.g., in the series 1, 2, $\cdots$ we bring the numbers 2, 3, $\cdots$ successively to the first place, leaving the rest unchanged, and in each resulting series do likewise for the second place, then similarly for the third, and so on, the whole set is well ordered by the process of formation.

4. By an *infinite permutation* is to be understood here a pairing whereby infinitely many objects change their places. Thus, for example

$$(1, 2) \circ (2, 1) \circ (3, 4) \circ (4, 3) \circ \cdots$$

is an infinite permutation of the series 1, 2, $\cdots$ yielding the new series 2, 1, 4, 3, 6, 5, $\cdots$. Another example is obtained by interchanging the powers of 2 with the corresponding powers of 3, giving the series 1, 3, 2, 9, 5, 6, 7, 27, 4, 10, 11, $\cdots$.

5. The polynomial elements that can be made with an ω-series of digits and an ω-series of indices form an enumerable set, since they are in one-to-one correspondence with the set of algebraic polynomials in which all coefficients are natural numbers. Let us order these polynomials according to the rules

$$(1) \qquad (i, m) < (i, n) \quad \text{if } m < n;$$

$$(2) \qquad (i_1, m) < (i_2, n) \quad \text{if } i_1 < i_2;$$

(3) $p < q$ if p has fewer terms than q;

(4) $p < q$ if they have the same number of terms but the first term of p that is not in q is lower than the first term of q which is not contained in p.

It is not difficult to show inductively that in this way the set is well ordered. Its type of order is the same as that which results from arranging the natural numbers according to their prime factors—first simple primes, then products of two primes, of three primes, and so on. This observation also proves at once that the set is countable.

6. The polynomial elements $\Sigma(i, n)$ are now to be used in forming infinite permutations. In the series 1, 2, $\cdots$ let the monomial (i, n) interchange an ω-series of products of n primes with a similar series of products of i primes, and let the polynomial $\Sigma(i, n)$ effect simultaneously the exchanges indicated by its several terms. Thus the first primitive (1, 1) is an identical transformation, leaving the series unchanged; the second monomial (1, 2) is to interchange the simple primes with some ω-series of products of two primes; let it be the first one, $2p_i$, and we get the permutation 1, 4, 6, 2, 10, 3, 14, 8, 9, 5, 22, 12, 26, 7, 15, 16, 34, 18, 38, 20, $\cdots$. The transformation numbered ω is (2, 1); let it exchange the simple primes and the series $3p_i$. The next transformation is (2, 2); let it interchange the series $2p_i$ and $3p_i$, yielding the series 1, 2, 3, 9, 5, 15, 7, 8, 4, 21, 11, 12, 13, 33, 6, 16, 17, 18, 19, 20, 10, $\cdots$. We have, then, an enumerably infinite set of infinite permutations, since to every polynomial has been assigned a finite set of transpositions as just described. Whether these transpositions are all used does not interest us.

7. It is now easy to obtain non-denumerable sets by repeating the above process. To illustrate, let (1, 2) operate on the series of primes in the result of the transformation (2, 2); it produces the series 1, 2, 7, 9, 13, 15, 3, 8, 4, 21, 29, 12, 5, 33, 6, 16, 43, 18, 19, 20, 10, 39, $\cdots$. Let $p_i^{(0)}$ denote a polynomial and $P_i^{(0)}$ the result of applying it as in § 6, where i is the ordinal symbol assigned to $p_i^{(0)}$ by the rules of § 5, and let the ω-series used in § 6 be numbered likewise. Then $P^{(1)} \sim (i, n)$ shall be the permutation obtained by applying $p_i^{(0)}$ as a transformation to the series whose ordinal symbol is n. The new symbols (i, n) are to be combined into new polynomials $p^{(1)}$ as in § 2 and ordered by the rules of § 5. The new set of permutations $P^{(1)}$ cannot be put into one-to-one correspondence with set $[P^{(0)}]$. For on this assumption there still exists a $P^{(1)}$ which differs from $P_1^{(1)}$ in the transformation $p^{(0)}$ performed on its first ω-series, differs at the same time from $P_2^{(1)}$ in the transformation performed on its second ω-series, and from $P_n^{(1)}$ in that performed on its nth ω-series (where n runs through the values 1, 2, $\cdots \omega, \cdots, \omega^2, \cdots$), and is therefore not contained in the assumed list, which is a contradiction. Hence $[P^{(1)}]$ is not enumerable.

8. It is worth while to study the preceding types of order in a different way, following Cantor; they are all comprehended under the concept of a "Belegung." In forming the primitive elements of the $p^{(0)}$, two places are covered with an ω-series of numbers; this covering is denoted by ω^2. For the binomials, four places i_1, n_1, i_2, n_2 are covered with an ω-series, indicated by ω^4. The polynomials of N terms have $2N$ places to cover, giving the ordinal symbol ω^{2N}, whence the whole set $[p^{(0)}]$ is of ordinal type ω^ω. And this is the same as the type of the set of ω-series of prime factors in the system of natural numbers, for the products of two primes exhibit a single ω-series of ω-series, $2p_i$, $3p_i$, $5p_i$, $\cdots$; in the products of three primes $p_{i_1} \cdot p_{i_2} \cdot p_i$ the successive ω-series are obtained by covering two places i_1 and i_2 with an ω-series, with four factors three places are covered, and so on. The same type again is exhibited in each $P^{(0)}$ (cf. § 5) and therefore in forming the $P^{(1)}$ we have likewise a covering. In the $P^{(1)}$ each ω-series may be re-arranged as an ω^ω-series and this process admits an ω-series of repetitions, but no more; beyond this point it yields nothing new.

There is a bit of formalism in the preceding which perhaps calls for closer scrutiny. The set of products of N primes is in the strict sense a covering of N factors with an ω-series of values. Think of a row of N boxes or spaces into each of which a compositor throws a prime. The totality of different ways in which he might set up this line is precisely what is meant by the symbol ω^N. The symbol ω^ω, then, would mean, by analogy, that to every one of an ω-series of spaces is assigned arbitrarily a prime number. But if an ω-series of places is covered in this way with the series of primes we certainly do not get the set of natural numbers. And the covering of an ω-series with an ω-series is not a "limit" of the coverings ω^N, because the set $[p^{(0)}]$ lies between them. The use of ω^ω for this set is merely a convenient symbolism not capable of supporting any logical deductions.

9. The process by which the permutations $P^{(1)}$ were obtained from the $P^{(0)}$, however, being carried out by finite arithmetical steps, is not liable to inconsistency and, moreover, can be repeated. Thus we get a new set $[P^{(2)}]$, the digit n running through the series of all ordinal values lower than the ordinal type of the set of series in $P^{(1)}$. Suppose that to every $P^{(2)}$ has been assigned a $P^{(1)}$. There exists a $P^{(2)}$ from which

every $P_\nu^{(2)}$ differs in its νth transformation (i_ν, n_ν) for values of ν forming a series of type $> \omega$. Hence the terms in the corresponding polynomial $p^{(2)}$ are a set whose type is $> \omega$. But there are no such polynomials. Therefore the $P^{(2)}$ cannot be put into one-to-one correspondence with the $P^{(1)}$. The same reasoning holds for the $P^{(1)}$, so we have two proofs that this set is non-enumerable.

10. The process of § 7 applied to the permutations $P^{(2)}$ yields a new set $[P^{(3)}]$ and the reasoning of § 9 shows that this set is not equivalent to $[P^{(2)}]$. Therefore by strict induction from N to $N + 1$ we infer the existence of an ω-series of sets of infinite permutations no one of which can be put into one-to-one correspondence with its predecessor. Ordinally $[P^{(N+1)}] > [P^{(N)}]$ for $N = 0, 1, 2, \cdots$. In Cantor's terminology, the set of infinite permutations of a simple infinity of objects presents an ordinal type higher than any finite aleph.

McPHERSON, KANSAS.

GEORGE WILLIAM HILL, 1838–1914.*

GEORGE WILLIAM HILL was the son of John William Hill and Catherine Smith, and was born in New York City on March 3, 1838. Both his father and grandfather were artists and he himself was of English and Huguenot descent. His early education like that of most of the men of his time in America gave him few advantages. In 1846, when his father moved from New York to the farm at West Nyack, the country was too busy with material development to produce many teachers who could give any but the most elementary instruction, and the country school which he attended must have been inferior in this respect to those of the larger cities. Even at Rutgers College in New Jersey, to which Hill was sent owing to the exhibition of unusual capacity and from which he took his degree in 1859, the course probably went but little beyond that now found in secondary schools. There, however, he came under the influence of a man whose ideas on education were unusual. Dr. Strong, according to Hill's evidence, believed only in the classic treatises; but little published after

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* Reprinted, by permission, from the *Proceedings of the Royal Society* May 3, 1915.

1840 was admitted to his library. Hill's sound knowledge of the fundamentals of his subject is doubtless due to this course of reading.

Hill's first paper, published in 1859 when he was but twenty-one years of age and before he had taken his degree at college, is a half-page note on the curve of a drawbridge. Two years later he showed his capacity in the essay which gained a prize offered by *Runkle's Mathematical Monthly* for the best solution of a problem connected with the constitution of the Earth. President R. S. Woodward, who has himself worked much at this subject, says that the memoir is still worthy of careful reading.

In the year 1861 he joined the staff of the Nautical Almanac office which then had its headquarters in Cambridge, Mass., and for a year or two he worked there and thus had an opportunity for association with some of the ablest men of the time in astronomical science. But he soon obtained permission to do his work at the home in West Nyack which he never seemed to leave willingly during the rest of his life. It was there that nearly all his best work was done. In fact he was only away from it for one considerable period and this is covered by his residence in Washington from about the year 1882 until 1892; even during that time the summers were generally spent in West Nyack.

In the first ten years after leaving college, Hill only published eight papers and none of them deal with celestial mechanics in the modern sense of the term. But from his output after that time it is evident that he had been reading and digesting the newer treatises and memoirs as they appeared. Delaunay's two magnificent volumes on the lunar theory were published in 1860 and 1869, respectively, and the methods of that investigator exercised a fascination over Hill for the rest of his life. The other great lunar theorist of the period, P. A. Hansen, had been explaining his methods for many years before this time and Hill was probably one of the few men of his time who understood them thoroughly. He does not seem to have been particularly drawn to them, although they are used in his theories of Jupiter and Saturn with but little alteration. It is difficult to find many traces of other influences in his work. His most celebrated memoir, it is true, is based on one of Euler's numerous methods, as he himself tells, but after the start he proceeds entirely on lines of his own devising.

The publications which follow his first attempts during this early period exhibit knowledge of theoretical astronomy and the power to handle large masses of numbers rather than any unusual mathematical ability. In his discussion of the observations of the great comet of 1858 which was undertaken to obtain a satisfactory orbit (1867), he has to deal with 363 places gathered from many sources. As usual with Hill, he does not confine himself to the main point but discusses systematic errors between different observatories and those due to the size of telescope used. His final conclusion is that there is no evidence of any force other than gravitation influencing the motion of the comet.

It is probable that his work on this body was responsible for the next three papers: on the reduction of star places, the determination of the elements of a circular orbit and the conversion of latitudes and longitudes into right ascensions and declinations, or at any rate that it drew his attention to these fundamental problems. But he was soon to lay them in the background for more original investigations in celestial mechanics proper. One can see in his published work the gradual approach to this subject. His tenth memoir is a correction to the elements of the orbit of Venus from observations extending over 33 years. It is followed by a derivation of the mass of Jupiter from the perturbations of certain asteroids, and the calculation of an inequality of very long period in the motion of Saturn. Shortly before, however, he had been assisting in the campaign which had started some years earlier to get the utmost out of the transits of Venus in 1874 and 1882. Part II of the Papers of the U. S. Commission relating to the transits is by his hand; it consists of charts and tables for facilitating predictions of the several phases at any place on the globe.

The active period of Hill's work in celestial mechanics began in 1872. Between that year and 1877, when his two chief memoirs appeared, he published eleven papers on various phases of the subject besides seven others in pure and applied mathematics and the long transit of Venus calculations already mentioned. Most of them are quite brief and call for no special mention.

In order that the value of Hill's contributions to celestial mechanics and more particularly to the lunar theory may be made clear, it is necessary to say a few words as to the condition of the subject at the time they were published. For two hundred years mathematical astronomers, many of them of the first rank, had been devoting their energies to furnishing a complete demonstration of the power of the law of gravitation to account for the motions of all the bodies in the solar system within the degree of accuracy of the observations. In the third quarter of the nineteenth century it was evident that this demonstration would soon be made. Leverrier was publishing his tables for the positions of the great planets, while Hansen and Delaunay had completed their work on the moon. For the purposes of navigation all needed accuracy had been obtained, and from the scientific side there seemed to be but few matters which needed explanation: the final polish which a few industrious workers might give was the last step. There was thus danger that the subject of celestial mechanics might encounter a blank prospect. The number of investigators began to dwindle. At the same time, pure mathematics and physics were showing vast territories to be explored, while the discovery of spectrum analysis and the use of the photographic plate attracted many astronomers who earlier would have devoted themselves to the mathematical side of the subject. From the old point of view this attitude on the part of astronomers was justifiable. But Hill saw that there were problems other than the mere verification of the law of gravitation by comparisons of theory and observation of the chief bodies in the solar system which would demand solution. He also saw, partly from the industrious work of Newcomb on the old and modern observations of the moon, that even the enormous labors of Hansen and Delaunay on the theory of its motion would demand extension and verification if a test of the Newtonian law to the degree of accuracy of the observations were required. For the former object, a new set of problems must be formulated and a start made towards their solution; for the latter, a new method of procedure was practically necessary, for it was almost certain that no one would repeat the calculations which appeared to have been pushed as far as was humanly possible with the adopted methods. These two sides of Hill's work are quite distinct even though they both start from the same memoir.

The older lunar theorists had taken the ellipse as a first approximation, that is, at the start the action of the sun was neglected. Hill proposed a first approximation in which a portion of the sun's action should be taken into account. If an examination of Delaunay's final expressions for the longitude, latitude, and parallax be made, it is seen that the infinite series proceed along powers of five parameters and that the rate of convergence along powers of one of these, the ratio of the mean motions of the sun and moon, is far more slow than along powers of the others, owing to the presence of large numerical factors. Hill conceived the idea of neglecting all these other parameters and then finding the series in powers of this ratio alone, with all needed accuracy. He set up the equations of motion, solved them and gave formulas of recurrence which enabled him to avoid the slow approximation methods which generally advanced the degree of accuracy by only one or two powers of the ratio at each step; in his method it advances by four powers of this ratio. The expressions are worked out both literally and numerically, the latter being taken to fifteen significant figures, a number not very much in excess of what is actually required.

As obtained, the coordinates are referred to axes which move with the mean velocity of the sun round the earth and in this form the expressions involve the time through its presence in multiples of a single angle. In the transformations which are necessary to convert rectangular coordinates to polars Hill makes full use of the method of "special values" or, as it is now called, of harmonic analysis and synthesis. He was always very fond of this kind of transformation, using it much in later years and even attempting to systematize its use when many hundreds of terms were present.

It would be unjust in this connection not to mention the indebtedness of Hill to Leonard Euler, probably the greatest of lunar theorists since Newton. Euler, as Hill remarks, had had the idea of starting the theory in the same way with moving rectangular axes and with the same first approximation and had carried it out to a considerable extent in his theory published in 1772 and in a later memoir.

The further steps outlined by Euler and quoted by Hill consist of the determination, step by step, of the terms arranged in powers of the parameters which had been neglected. Each step is to consist of the complete calculation with all needed accuracy of the function of the time and the ratio of the mean motions which multiplies each combination of powers of the remaining parameters. There are several difficulties in following this process. The chief one, which Hill solved in the memoir on the perigee of the moon, is the determination of the first new angle containing the time which arises in the second approximation. In later approximations this angle also involves all the parameters and other methods are needed to find the new portions depending on them. Euler possibly foresaw this; Hill certainly did, but he never carried his work to the degree of approximation which would need them. The method has been used by the writer for the construction of a complete theory of the moon's motion.

The expressions for the coordinates, referred to the moving rectangular axes, have another property: they form Fourier series and are therefore periodic. The resulting orbit in this moving plane is consequently closed. Recognizing this fact, Hill draws the curve. But he saw that the orbit was of interest apart from its application to the lunar problem, for he immediately proceeds to trace, with some care, orbits for values of the ratio of the mean motions other than that which holds for the actual moon and sun. He thus obtains a family of such orbits. It is Hill's idea of the periodic orbit which, developed chiefly by Poincaré and G. H. Darwin, has given new life to the whole subject of celestial mechanics and has induced many mathematicians to investigate on these lines. The treatise of the former, Les Méthodes nouvelles de la Mécanique céleste, is based mainly on this idea. Darwin actually traced many such orbits under varying conditions.

There is still another portion of this memoir which has been largely used as a foundation for investigations into the stability of celestial systems. If the eccentricity of the earth's orbit round the sun be neglected, it is possible to write the relative energy equation in a finite form. Referred to the same moving axes, the square of the velocity can, in fact, be expressed as a finite algebraic function of the coordinates. Since the square of the velocity can never be negative, this function, equated to zero, gives the equation to a surface which the moon cannot cross. As the surface consists of various ovals and folds, we can obtain certain limitations on the path of the moon and therefore carry forward the question of the stability of its motion one important step. Hill draws the surfaces for a limited case. Darwin made extensive use of a similar diagram for a more extended case, and many others have followed on the same lines.

Thus this memoir of but fifty quarto pages has become fundamental for the development of celestial mechanics in three different directions. Poincaré's remark that in it we may perceive the germ of all the progress which has been made in celestial mechanics since its publication, is doubtless fully justified. It has sometimes been said that Hill did not appreciate at the time the importance of his work. Hill was far too modest about his own achievements to lay any such stress on his productions as has the scientific world. But it does not require an extended study of his memoirs to see that his vision often went beyond the particular matter in hand.

The second memoir of 1877, "On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon," has already been referred to. It is essentially a continuation of that part of the researches which deals directly with the lunar problem, although published a few months earlier. While not so far reaching from the point of view of future developments, it is even more remarkable as an exhibition of Hill's powers of analysis. In it, the determinant with an infinite number of elements is raised from a nebulous possibility to an instrument of computation. Hill's periodic orbit contained only two of the four arbitrary constants which the complete solution of his differential equations requires. He therefore proceeds to find an orbit—no longer periodic—differing slightly from the periodic orbit but still satisfying the differential equations to the first power of the small variation. The equations obtained are two of the second order and linear with respect to the two unknown dependent variables. An able analysis with the use of known integrals, enables him to reduce the solution to that of one of the second order in the normal form

$$\frac{d^2p}{dt^2} + Vp = 0,$$

where V is a known Fourier series depending on the time. Knowing the form of the solution

$$p = \Sigma a_i \cos \{c(t - t_1) + 2i(n - n')(t - t_0)\}$$

from previous work in the lunar theory, a form he justifies by general considerations, Hill substitutes and obtains an infinite series of linear equations for the determination of the unknowns a_i. But c is also unknown and it does not enter in a linear form. The a_i are eliminated by means of a determinant with an infinite number of rows and columns equated to zero; this is therefore a determinantal equation to find c, the main object of the investigation.

Then follows a remarkable series of operations. The determinant is reduced to a convergent form (though it was left to Poincaré to furnish the proof of convergence) by dividing each row by a suitable factor which reduced every element of the principal diagonal to unity. Next, the unknown c must be isolated; Hill achieves this by recognizing that if c be a root so must $c + 2i$ be also a root and that therefore the roots can be expressed by a cosine function. On the assumption that there are no other roots, he equates the determinant to the cosine function, obtaining the constant by comparing the highest (infinite!) power of c on each side of the equation. A particular value of c (not a root) can be inserted in the identity thus obtained. In this way, Hill reduced the work to the computation of an infinite determinant every element of which is known. He gives a general method for this expansion which enables him to tell at once the order of the terms neglected when the series is cut off at any place. Each term of this series, however, consists of singly, doubly, . . . infinite series which must be summed. The labor at this stage was very great and it caused a corresponding liability to error. Hill carried it through with complete success in its general form, afterwards substituting numbers and determining c to sixteen significant figures. The principal part of the motion of the moon's perigee is immediately deducible from c. President Woodward relates that the determinant was solved during one of two trips which Hill made to the northwest region of Canada; I imagine, however, that this statement refers to the method to be adopted rather than to the actual computation.

The story of these two memoirs is incomplete without a notice of the work of J. C. Adams on somewhat similar lines. Almost immediately after their publication, a brief paper by him appeared in the *Monthly Notices* of the Royal Astronomical Society. He had also taken up Euler's idea and had obtained the variation orbit as a first approximation. But he turned to the motion of the node instead of to that of the perigee. The investigation here follows lines very similar to those of Hill, the solution of the infinite determinant being closely analogous.

It is convenient at this stage to take up Hill's work rather by subject than in chronological order. The periodic orbit used with such excellent results in the lunar theory is tried later (1887) on the motion of the satellite Hyperion as disturbed by Titan and the results applied in a following paper to obtain the mass of the latter. These were written before the publication of Poincaré's researches. Only on one occasion did Hill make it the subject of a theoretical research and it was then probably stimulated by reading Poincaré's Mécanique céleste. As the title, "Illustrations of periodic solutions in the problem of three bodies," indicates, it consists of applications to certain bodies in the solar system.

From time to time a paper was published advancing the applications to the lunar theory. In one, the periodic orbit is extended so as to include the terms which depend on the ratio of the parallaxes of the sun and moon as well as on the ratio of the mean motions. In another paper the terms dependent on the latter ratio and on the first power of the solar eccentricity are computed. In still another paper he calculates the expression for the principal part of the motion of the moon's perigee as far as m^{11} literally in order to settle the correctness of Delaunay's value, which had been questioned as to certain of the earlier powers of m by Andoyer. Beyond these, he seems to have made no effort to continue the work in this direction. Possibly this was due to the heavy labor on the theories of Jupiter and Saturn which engaged him at least until 1892. In fact, as early as 1888 he stated in a letter to Sir George Darwin that he scarcely expected to proceed with the subject.

His fondness for Delaunay's methods has already been mentioned. One of his most valuable memoirs is an application of them to the calculation of the smaller perturbations of the moon's motion which arise from the action of the planets and the figure of the earth. Hill, using Delaunay's methods and results, showed, in a short paper on the Jovian evection, that the whole action of the earth and sun on the moon could be treated as known from the start and that therefore only one approximation was needed in order to get the effect of any disturbance whose square could be neglected. All later investigators have used this method. The formulas of Delaunay are literal, while Hill's final equations for the calculation of the effect of any small disturbing force have the great advantage of well-determined numerical coefficients to be multiplied by the constants which depend solely on the nature of the given force.

In an earlier paper he had also shown how the disturbing function for direct planetary action can be expressed as a series of products, one factor in each product containing the coordinates of the earth and moon only, while the other contained those of the earth and planet only. The former could therefore be computed once for all; it was the latter which required separate computations for each planet. This paper has also formed the basis for all the complete calculations of the planetary disturbing forces which have been made since its publication in 1883.

But Hill's most extensive application of Delaunay's theory is made in its original form to the calculation of the inequalities produced by the figure of the earth. While he carried these to the degree of accuracy needed for observation, the method appears to be somewhat long and complicated. It has to be applied in a literal form and this requires expansions which converge very slowly. As a matter of fact a few days work with the methods which he adopted for the planetary terms will furnish the inequalities with all needed accuracy. In the first part of this paper Hill, not content with the values for the flattening of the earth which were then in use, deduced one directly from a large number of pendulum observations all over the earth. The result, 1/288, is considerably larger than most of the other determinations and notably so than that of Helmert, 1/298, deduced from the same class of observations. The memoir occupies over a hundred and forty pages and must have demanded an enormous amount of careful and accurate algebraic computation. To complete the account of his work on the lunar theory, mention must be made of his calculation, by de Pontécoulant's method, of the principal inequalities produced by the motion of the ecliptic. Hansen was the only writer who had found the term in longitude as well as in latitude, and nearly all his calculations of the small perturbations are doubtful. Hill, of course, obtained correct results as far as he went in the matter.

Newcomb, who had taken charge of the American *Ephemeris* in 1877, soon induced Hill to undertake the theories of Jupiter and Saturn, and so give material assistance in his plan of forming new tables of the planets. The method adopted is that of Hansen with only a slight modification which consisted in expressing the computations directly in terms of the time instead of using two auxiliary angles. That he used an old method in preference to devising a new one is perhaps unfortunate even though the result leaves little to be desired. Had he taken more time over the preliminary stages we should probably have had something new and original, for Hill was then at the height of his powers as a mathematician. But he was doubtless under some pressure from Newcomb, who wished to complete his great plan during his tenancy of the office of director, and Hill himself may have desired to finish the calculations as soon as possible in order that he might return to West Nyack. However this may be, he completed the task successfully as may be judged from the small residuals which he obtains after a comparison with observations extending over 150 years. The tables which he formed from the theories of the two planets are now used in most of the national ephemerides.

In 1882 Hill published a memoir of some length on Gauss's method for computing the secular perturbations of the planets. Gauss had outlined only the general idea. Hill takes it up and develops in detail the formulas to be used. In the course of the work he finds that a considerable portion of the calculation depends on three elliptic integrals which may be needed for values of the argument up to 50°. Consequently a large part of the paper consists of the tabulation of these to eight places of decimals at intervals of a tenth of a degree; the first and second differences are also printed so that the tables are in form ready for interpolation. As an example he computed the secular perturbations of Mercury by Venus with great accuracy. Two further papers on the same subject appeared in 1901.

In these years Hill published a number of short papers in the *Analyst*, a journal no longer in existence. Sometimes they are merely solutions of well-known problems, at other times, simplifications of proofs of theorems which had evidently presented difficulties to him and which he felt needed elucidation or elaboration—two favorite words with him. But Hill was not a great expositor: even for those familiar with the subject his work is often difficult and sometimes obscure. Newcomb used to say that if Hill had only the faculty of explaining his own ideas he might have avoided many an error and saved much time. Hill's ability to assimilate and extend the work of his predecessors, at any rate in his earlier days, doubtless prevented him from appreciating the difficulties of others. When the reader is used to Hill's style of composition and his general plans in writing out what he had to say, his arguments are much more easily grasped, but he is rarely anything else than concise.

In his last years Hill still continued to publish in spite of failing health. He covered a variety of topics, several of them quite away from the region of celestial mechanics. One of the most extensive of his papers is a memoir on dynamic geodesy, the last in the fourth volume of his collected works and not previously published elsewhere. Some later papers on a variety of subjects will appear in a fifth volume, to be published, like the previous four volumes, by the Carnegie Institution of Washington.

If an attempt is made to regard Hill's work as a whole and to try to find out his point of view, one thing stands out clearly: a desire to obtain exact knowledge about natural phenomena, in however limited a field, which could be expressed in a numerical form. He never seemed to hesitate about making long calculations and apparently had a positive liking for obtaining his results to many places of decimals. But unlike the tendencies of those who engage much in computation, his mind did not seem to get cramped by figures. Not only could he see both trees and wood, to adopt a familiar simile, but could trace paths in the wood and keep his eyes open for roads which led in directions other than that he was exploring. He had remarkable ability for algebraic manipulation which reached its highest manifestation in the memoir on the perigee of the moon. The more modern sides of mathematics appealed to him but little; if a formula or a series could be reduced to numbers, such questions as convergence did not trouble him much, a point of view which has later been fully justified by Poincaré. He seemed to take but little color from the work of others. Even when, as in many cases, he starts with the results of some previous investigator his writing shows only slight influence of the source of his ideas; it is individual and carries the reflection and methods of his own mind.

Hill never married. He lived much alone, but while resident in Washington would take long walks on Sunday, often with one or two companions. He was fond of botany without being a collector of specimens and found his chief outdoor recreation in the study of nature. He made two long canoe trips in the northwest of Canada. A carefully written diary illustrated with photographs of the second expedition which took him by rivers and lakes from Lake Superior to Hudson's Bay, is amongst the books which he left in his will to Columbia University.

He was president of the American Mathematical Society from 1894 to 1896, and served as lecturer on celestial mechanics in Columbia University from 1898 to 1901. The manuscript of his lectures shows that they must have cost him much labor; it contains long algebraic developments and is apparently intended to be a more or less complete account of the methods by which the motions of the moon and planets are calculated. His numerous honors include foreign membership in the Royal Society, the Paris Academy, and the Belgian Academy. He received the Schubert Prize (Petrograd), the Damoiseau Prize (Paris), the Gold Medal of the Royal Astronomical Society and in 1909 the Copley Medal of the Royal Society.

His chief characteristic was a single-minded devotion to the subject which he had made his own. A highly sensitive conscience was always apparent in his dealings with the world: one year he refused to accept the salary of his lectureship at Columbia because no students then appeared to attend the course, and this in spite of the fact that the endowment left him absolutely free to lecture or not as he chose. In later years, he rarely left West Nyack, owing to ill health. He died on April 16, 1914, from heart failure and was buried near the graves of his ancestors not far from his home.

E. W. Brown.

DICKSON'S LINEAR ALGEBRAS.

Linear Algebras. By L. E. Dickson, Ph.D. (No. 16, Cambridge Tracts in Mathematics and Mathematical Physics.) Cambridge, University Press, 1914. 8vo. viii + 73 pages.

> And still they read, and still the wonder grew,
> That one small tract contain so much. . . .

A substantial and systematic introduction to general linear algebras, associative and non-associative, a revision of Cartan's theory of linear associative algebras over the field of complex numbers, the results of Wedderburn's theory of such algebras over a general field, the relation of linear algebras to finite and infinite groups and bilinear forms, the consideration of various special algebras and a wealth of historical and bibliographical references in footnotes—in seventy-three pages!

The space at the author's disposal was doubtless limited. To have compressed so much material into it is indeed an achievement. With such an end in view the author could not afford to bandy words. He must needs make a point of being brief. And he has made this point well. His sentences are concise, and more often than not are condensed by the use of symbols rather than words or, now and then, by some ingenious form of brevity. The neatest of these is the use of (3_1), (8_2), etc., to denote "the first of the formulas (3)," "the second of the formulas (8)," etc. The equality sign is called upon to do duty in various rôles: thus, to denote the equivalence of two references, "*Proc. London Math. Soc.*, . . . (= *Coll. Math. Papers*, . . .)"; or to indicate the coordinates of a point, "the point $C = (c, d)$."

In attempting to attain to brevity, an author runs the danger of sacrificing clearness. Doubtless it is to avoid this pitfall that some authors spare no space in inserting every last bit of reasoning in their proofs and discussions. But a text-book from the pen of such an author affords the reader merely mental entertainment, whereas, in the reviewer's opinion, the ideal text-book on any except the elementary subjects should also require of the reader mental exercise. And often the more vigorous the exercise required, provided it remains within the possibilities of the reader, the more really lasting and beneficial is the entertainment.

Needless to say, the book under review is of the entertainment-plus-exercise type. The author demands continually of the reader that he supply bits of reasoning. But he is always very careful to make adequate suggestion as to their nature, usually by reference to previous sections and formulas. Thus actual clarity does not suffer; there is not an obscure place in the tract if one reads every word at its full value. Especially is the author to be complimented on his able revision of the complicated theory of Cartan in Part II.

For a complete understanding of a book, a reader needs not only to master the details but, as he reads, to strip them of their unessentials and assign them to their proper place in a skeleton structure of the whole subject, so that in the end he will see the aims, methods and facts, their significance and relative value, in a true perspective. Naturally he must do most of this digestive work himself, but the author may furnish him invaluable help by appropriate comments now and again. To do this, and do it well, is no easy task at the best. Still it is simpler, certainly, in a book of the type under review, in which the abstraction from the unessentials is already partly done, than in one of the purely entertainment type, in which such abstraction is rendered exceedingly difficult by the great mass of detail. But the present author makes no use of this advantage; in fact he makes next to no attempt to keep the reader properly oriented. As a result, his book presents always one and the same aspect, a desert of statement and proof with not a refreshing oasis in sight, where the reader may pause to rest and take account of stock. An amount of space, equivalent, say, to a half dozen unbroken pages, devoted to "oases" distributed judiciously throughout the book, would, in the reviewer's opinion, improve it greatly in raising the quality of clearness of the whole up to the standard of that attained in the details.

The tract is divided into four parts. Part I contains "definitions, concrete illustrations and important theorems capable of brief and elementary proof." It deals with the general linear algebra, associative and non-associative. Now the associative algebras of real interest are those possessing principal units, for in such an algebra without a principal unit division, if possible at all, is not unique. But with the introduction of non-associative algebras into the discussion, the presence or absence of a principal unit takes on new meaning. Hence in reading this first part, the beginner should keep an eye peeled for the appearance and disappearance not only of "associative," but also of "principal unit." In Part II we find the heart of the general theory, presented from Cartan's point of view for linear associative algebras over the field of complex numbers. The author chooses Cartan's treatment rather than those of Molien, Wedderburn, or Frobenius, because it alone remains within the bounds of linear algebras for its methods and proofs. The other treatments depend in whole or in part on the theory of groups or bilinear forms, and it is the purpose of the writer not to appeal to these to establish his results. The relations of these allied theories to linear algebras are adequately discussed, however, in Part III. Part IV contains the statement and elucidation of Wedderburn's results concerning linear algebras over an arbitrary number field. Numerous examples, illustrative of the theory, are given in smaller print throughout the book.

Part I begins with the development of the system of ordinary complex numbers. We are disappointed to find that the method of procedure here does not wholly conform to that used later in developing the system of general hypercomplex numbers. In this latter method the n-tuple form $(x_1, x_2, \cdots, x_n)$ of the hypercomplex number x is discarded just as soon as the processes, addition and multiplication by a scalar, requisite to obtain the more useful form $x = e_1x_1 + e_2x_2 + \cdots + e_nx_n$, where $e_1, e_2, \cdots, e_n$ are the n units, have been defined. But in the development of the ordinary complex numbers (and also in that of the matric algebra which follows in §§ 3, 4) there is no explicit mention made of multiplication by a scalar, and the couple form (a, b) of the complex number is retained almost to the end.

In § 2 a set of complex numbers is defined as a number field "if the rational operations can always be performed unambiguously within the set." The reviewer has racked his brain in a vain attempt to conceive a reason for the insertion of "unambiguously." If a rational operation on ordinary complex numbers—the definition can refer only to these, since no others have as yet been defined—is performable at all, i. e., if it is not division by zero, then the result is certainly unique.

The author adheres rigorously to the use of the concept of number field throughout. Thus, he defines (§ 5) as a linear algebra* over the field F the set of all numbers $\sum_{i=1}^{n} x_i e_i$, with coordinates x_i in F, subject to combination by suitably defined addition, subtraction, multiplication by a scalar in F, and by a distributive multiplication with the multiplication table $e_ie_j = \sum_{k=1}^{n} \gamma_{ijk}e_k$ $(i, j = 1, \cdots, n;\ \gamma\text{'s in } F)$.

In such an algebra, right-hand division, i. e., solution of $xx' = y$ $(y \neq 0)$ for x', is possible and unique if and only if the right-hand determinant of x, $\Delta(x) = |\sum_{i=1}^{n} x_i\gamma_{ijk}|$ $(j, k = 1, \cdots,$

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* From now on in this review "algebra" will be used in the sense of "linear algebra."

n), is not zero. The left-hand determinant $\Delta'(x)$ is similarly related to left-hand division (§ 6). The number ϵ, if it exists such that $x\epsilon = \epsilon x = x$ for every number x of the algebra, the author sometimes calls "principal unit," but just as often "modulus," a term aptly and justly described by Study as "den auf alles Mögliche angewendeten Ausdruck 'Modul.'"[*] A necessary condition that there exist a principal unit is that both kinds of division be, in general, possible. The condition is proved sufficient if the algebra is associative, but for a non-associative algebra the author does not enlighten us (§ 7).[†]

In § 8 linear transformation of the units of an algebra is considered and the equivalence of two algebras under such linear transformation defined; in §§ 9, 10 it is noted that the general number x of an associative algebra is the root of an equation of degree $\leqq n + 1$ and satisfies any algebraic identity in an ordinary complex variable made up of functions without constant terms, or if the algebra moreover contains a principal unit, is a root of an equation of degree $\leqq n$ and satisfies *any* algebraic identity in an ordinary complex variable. By application of these results the author shows, in § 11, that every number of an associative algebra over the field $F(a)$ of real numbers, which has no nilfactors (divisors of zero), is a root of a quadratic equation, and hence obtains a neat proof that the only such algebras are $F(a)$ itself, the field $F(i)$ of ordinary complex numbers, and the algebra of real quaternions. In § 12 we find a discussion of the simplest properties of real quaternions, in § 13 a proof that the complex quaternion algebra and the complex matric algebra with four units are equivalent, though their real sub-algebras are not. In § 14 the author represents Cayley's eight unit algebra, a non-associative generalization of real quaternions, as a "quasi-binary algebra with real quaternion coordinates" and thus gives simple and pretty proofs that this algebra contains no nilfactors and that in it, too, the norm of a product equals the product of the norms.

We now return to the general theory and introduce the right and left-hand characteristic determinants and equations $\delta(\omega)$, $\delta'(\omega)$, $\delta(\omega) = 0$, $\delta'(\omega) = 0$, where $\delta(\omega)$ and $\delta'(\omega)$ are derived from $\Delta(x)$ and $\Delta'(x)$ by subtracting ω from each element of the principal diagonals: The general number of an associative algebra is a root of both $\delta(\omega) = 0$ and $\delta'(\omega) = 0$. Suitable variations of this theorem for an algebra with a principal unit and for an arbitrary algebra are given. In § 17 a function of the x_i and the γ_{ijk} which reproduces itself under an arbitrary linear transformation of units with a factor depending only on the coefficients of the transformation is called a covariant (invariant, if it involves only the γ's). The determinants $\Delta(x)$, $\Delta'(x)$, $\delta(\omega)$, $\delta'(\omega)$ are absolute covariants of the general algebra. The proof given in § 16 of this is incomplete,[*] since the statement "let $\Delta(x)$ be not identically zero, so that there exists a modulus" assumes the algebra associative and hence leaves the theorem unproved for non-associative algebras for which $\Delta(x) \not\equiv 0$. The theorem is applied in § 18 to obtain the three (two) types of non-equivalent binary algebras over $F(a)[F(i)]$ with a principal unit.

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[*] Encyklopädie der math. Wiss., I, A, 4, p. 162.

[†] It is a simple matter to construct an example to show that the condition is not sufficient for a non-associative algebra.

The rank r of an associative algebra is the least positive integer such that x^r is a linear combination of lower powers of x whose coefficients are rational functions of $x_1, \cdots, x_n$ with coefficients in F. The corresponding equation $R(x) = 0$ is called the rank equation. If the algebra has a principal unit, $R(\omega) = 0, \delta(\omega) = 0, \delta'(\omega) = 0$ all have the same distinct roots. As an application, the author gives Study's classification of complex ternary associative algebras with a principal unit into five non-equivalent types according to the five possibilities for the degree and the multiplicity of the roots of the rank equation (§ 20).

The author defines an associative algebra with n units and a principal unit as reducible in its field F if it contains n numbers $e_1, \cdots, e_p; E_1, \cdots, E_q$, linearly independent with respect to F, such that $e_iE_j = E_je_i = 0$, $(i = 1, \cdots, p;\ j = 1, \cdots, q)$, and proves that such an algebra is the direct sum of two algebras.* He then sets up Scheffer's criterion for reducibility. In § 22 the direct product of any two algebras is defined and Wedderburn's generalization of Scheffer's theorem identifying an associative algebra, which has the quaternion algebra as a subalgebra and its principal unit as principal unit, as the direct product of the quaternion algebra and a second algebra.

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[*] The author's proof goes astray in assuming the existence of a principal unit. But the only use he makes of this assumption is to show, after he has proved that $D(X) = c\Delta(x)$, where c depends only on the coefficients c_{ij} of the transformation of units, that $c = 1$. Now, by inspection, c is the quotient of a homogeneous expression of degree $2n$ in the c_{ij} by $|c_{ij}|^2$. Since c cannot vanish unless $|c_{ij}|$ vanishes—otherwise we could exhibit values for the x_i, γ_{ijk} for which $\Delta(x) \neq 0$, while $D(X) = 0$—and $|c_{ij}|$ is irreducible, $c = k|c_{ij}|^l$, where k is independent of the c_{ij}. But evidently $l = 0$, and, by applying the identical transformation of units, $k = 1$.

Professor Dickson's proof holds for any algebra with a principal unit. In a letter to the reviewer he points out that the case of an algebra without a principal unit may be treated by enlarging it to an algebra with a principal unit by adjoining a new unit (as in the proof of Theorem 3, § 15), and also calls attention to a later proof by Miss Hazlett, *Annals of Mathematics*, vol. 16 (1914), p. 2.

In the last main paragraph (§ 23) of Part I a set of normalized units for any algebra is developed. Since their main purpose is to form a point of departure for Part II, we have need to describe them here only for an associative algebra with a principal unit ϵ. Suppose that the solutions of $\delta(\omega) = 0$ for a particular number a of the algebra are $\omega_1, \cdots, \omega_h$ of multiplicities $m_1, \cdots, m_h$, and consider one of these solutions, say ω_1. Now $a - \omega_1\epsilon$ is a nilfactor, that is, there exist solutions $y \neq 0$ of $(a - \omega_1\epsilon)y = 0$ or $ay = \omega_1 y$; take a complete system of solutions y, say t in number, as the first t of the new units $e_1, e_2, \cdots$. If $t < m_1$, there exist solutions z of $az = \omega_1 z + \sum_{i=1}^{t} c_ie_i$; take a complete system of solutions z, say t_1 in number, for the next t_1 of the new units. If $t + t_1 < m_1$, repeat; we will finally obtain m_1 numbers $\alpha_1, \alpha_1', \cdots, \alpha_1^{(m_1-1)}$, such that $(a - \omega_1\epsilon)\alpha_1^{(i)}$ is a linear function of the preceding α_1's. Similarly, there exist, corresponding to the root ω_j, m_j numbers $\alpha_j, \alpha_j', \cdots, \alpha_j^{(m_j-1)}$, such that $(a - \omega_j\epsilon)\alpha_j^{(i)}$ is a linear function of the preceding α_j's. The $n = m_1 + m_2 + \cdots + m_h$ new units thus obtained are linearly dependent; they are termed a set of units normalized relatively to the number a.

Part II consists of a "revision of Cartan's general theory of complex linear associative algebras with a modulus." In § 25 units having a character are developed. Denote the principal unit, expressed in terms of the units of § 23, by $\epsilon = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_h$, where ϵ_i is a linear combination of the α_i's. (The ϵ_i are called the partial moduli.) Then the number η is said to have the character (j, k) if $\epsilon_j\eta = \eta$, $\eta\epsilon_k = \eta$, and every other combination of η with an ϵ_i by multiplication is zero. Now $\epsilon_j\alpha_j^{(i)} = \alpha_j^{(i)}$, $\epsilon_k\alpha_j^{(i)} = 0$ and in particular $\epsilon_i^2 = \epsilon_i$,

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* He might have generalized the definition by demanding, instead of the existence of a principal unit, the existence of two numbers e, E, where e (E) is a linear function of the $e_i (E_i)$ with coefficients in F, which is not a nilfactor with respect to all other such linear functions, and the proof would still hold with but slight variation.

$\epsilon_i\epsilon_j = 0$; furthermore there exist m_j linearly independent linear combinations of the $\alpha_j^{(i)}$, such that the product of any one of them into a certain ϵ_i gives itself, and its product into any other ϵ_i gives zero, i. e., such that they have definite characters $(i, \cdot)$. Hence we have obtained n new units $\epsilon_1, \epsilon_2, \cdots, \epsilon_h, \eta_1, \cdots, \eta_k$ each having a definite character. We wish further to show that we can choose the η's so that each is nilpotent, i. e., so that some power of it is zero. Now the square of every η of character (i, j), $i \neq j$, is zero, since the product of a number of character (i, j) by a number of character (k, l) is zero, if $j \neq k$. And since the sum or product of two numbers of like character, if not zero, is of that character, all the numbers of character (i, i) of the algebra form a sub-algebra Σ_i; it is possible to choose the units of this sub-algebra, other than ϵ_i, as nilpotent numbers. Thus we have arrived at a set of normalized units consisting of the partial moduli and $n - h$ nilpotent numbers each having a definite character. We shall refer to such a set of normalized units as being of type K.

We now separate algebras into two categories, according as the determinant $\Delta(x)$ does not or does contain a non-linear irreducible factor. To prove the principal theorem for algebras A_1 of the first category, i. e., that for such an algebra a set of units of type K can be found having the further property that $\eta_i\eta_j$ is a linear function of those η's whose subscripts exceed i and j and have the same character as $\eta_i\eta_j$ (property L), the author shows that the nilpotent numbers of A_1 form a sub-algebra N, for which units η of definite characters enjoying the property L can be found. Conversely, if an algebra is given possessing a set of units of type K enjoying the property L, the γ_{ijk} of the multiplication table are determined (except for the non-vanishing ones in the $\eta_i\eta_j$) and inspection shows that $\delta(\omega) = (x_1 - \omega)^{k_1} \cdots (x_h - \omega)^{k_h}$, that is, that the algebra is of the first category. An algebra such as N is called nilpotent.

To every irreducible factor of the characteristic determinant for an algebra A_2 of the second category corresponds a sub-algebra of A_2, just as in the case of an algebra A_1. But here at least one of these irreducible factors is not linear. Each sub-algebra corresponding to such an irreducible factor contains linear combinations of nilpotent numbers which are not nilpotent. These numbers are peculiar to algebras A_2

and form the basis of Cartan's treatment of them. The details of the treatment are too intricate to permit of an adequate description in a brief space. We content ourselves with a statement of the principal results. First, any algebra A_2 contains a sub-algebra equivalent to a matric algebra with more than one unit. Hence Cartan's classification of algebras into those of the first and second categories coincides with that of Scheffers into those without and with quaternion sub-algebras. As the final result we have a process by means of which the algebras A_2 may be derived from the algebras A_1: the general number of an A_2 is obtained from the general number of an A_1, expressed in terms of units of type K having the property L, " by regarding the coefficient of ϵ_i to be a square matrix of p_i^2 elements and that of η_l, of character (i, j), to be a rectangular matrix of p_i rows and p_j columns, these matrices to be regarded as commutative with each ϵ and η." Thus we obtain from the units of an A_1, the multiplication table and the characteristic determinant corresponding to them, the units, multiplication table and characteristic determinant of an A_2.

In §§ 49, 50 the composition of an algebra with respect to the presence or absence of invariant sub-algebras is discussed. A sub-algebra I of an algebra A is termed invariant if the product of any number of I and any number of A in either order is a number of I. An algebra having no invariant sub-algebra is called simple and one having no nilpotent invariant sub-algebra semi-simple. A general algebra, A_1 or A_2, is the sum, but not necessarily the direct sum, of a semi-simple algebra and a nilpotent invariant sub-algebra. A semi-simple algebra, if not simple, is the direct sum of simple algebras and conversely. Finally, a simple algebra is matric, unary if an A_1, non-unary if an A_2, and conversely.

In concluding Part II the author shows that a commutative algebra must be an A_1 and, in particular, an A_1 which is the direct sum of its sub-algebras $\Sigma_1, \cdots, \Sigma_h$.

Part III begins with a consideration of the correspondence between associative algebras and linear groups. To each number y of any algebra correspond two linear homogeneous transformations (y), $[y]$ of the variables $x_1, \cdots, x_n$ into the variables $x_1', \cdots, x_n'$, defined by the equations obtained from the products $x' = xy$, $x' = yx$. If the algebra is associative, the set of transformations (y) is closed and so is the set of transformations $[y]$. If further the algebra contains a principal unit, there corresponds to it in each set the identical transformation, and hence the inverse of any transformation corresponding to a non-nilfactor exists (and belongs to the same set). Thus the two sets of transformations (y), $[y]$, corresponding to the non-nilfactors y, form groups G, G'. These groups are simply transitive, mutually commutative (reciprocal in the sense of Scheffers), and each is its own parameter group. Conversely, any two simply transitive reciprocal groups of linear transformations can be transformed by the same change of variables into a pair of groups G, G', defined by an associative algebra with a principal unit. A classical example of the representation of linear transformations by hypercomplex numbers is then given: if q and Q are variable and q_1, q_2 given quaternions, then $Q = q_1\, q\, q_2$ represents the 7-parameter group of transformations of similitude in R_4 leaving the origin invariant; the group represented by the equation when, in particular, q_1 and q_2 are conjugate and q and Q vectorial quaternions, is not the corresponding group in R_3, as the author states, since it does not contain the stretchings from the origin but only the rotations about it.

Section 53 shows that the theory of bilinear forms is equivalent to that of matrices and also, in the last analysis, to that of associative algebras; section 54 explains the relations between finite groups and their group matrices to linear algebras; and section 55 treats of Dedekind's idea of considering the relations giving the values of the products e_ie_j $(i, j = 1, \cdots, n)$ of the units of a commutative associative algebra as ordinary algebraic equations in the units as unknowns.

The decomposition of a linear associative algebra over a field F within this field is precisely the same as that given above (§ 50) for a complex associative algebra within the field $F(i)$, except that instead of the last statement we now have: any simple algebra over F is the direct product of a division algebra and a simple matric algebra, each over F, and conversely. A division algebra is one without nilfactors. The relation of the particular case, $F = F(i)$, to the general one is made clear by the fact that the only division algebra over $F(i)$ is $F(i)$ itself. Further, if F is the field $F(a)$ of all reals, the only division algebras, besides $F(a)$, are $F(i)$ and the algebra of real quaternions. Now, since it can be shown that a commutative matric algebra has a single unit, it follows that any real commutative algebra without nilpotent numbers is the direct sum of unary algebras equivalent to $F(a)$ and binary algebras equivalent to $F(i)$. Weierstrass showed that a real commutative associative algebra, possessing a principal unit and such that the only algebraic equations with an infinitude of the numbers of the algebra as roots are those in which the coefficients are multiples of one nilfactor, is equivalent within $F(a)$ to just such a sum of real unary and binary algebras and conversely.

In §§ 58–60 are given examples of various types of division algebras previously published by the author. In § 61 results concerning difference algebras, the composition, difference and reduction series of an algebra are given, and, in particular, Wedderburn's theorem to the effect that an associative algebra over F can be decomposed in one and but one way into the direct sum of irreducible algebras with principal units and, if the given algebra has not a principal unit, an algebra without a principal unit. The closing section (§ 62) of the tract contains brief mention of the work of Berloty, Scheffers, and Hausdorff on analytic functions of hypercomplex numbers.

Attention is called to the following misprints: page 4, line 8, read *of* instead of *by*; page 24, last line, $\xi^2-2L\xi$ instead of $\xi^2+2L\xi$; page 32, last line, e_5 instead of e; § 26, last line, $e_3\epsilon_1$ instead of $e_3\epsilon_4$; § 33, third line, $-\omega^5$ instead of ω^5; § 33, fifth line, omit "the" before "normalized units"; page 43, "(71)" applies to both equations; page 48, ninth line from bottom, insert "variables" before [2, 2]. In § 6, the condition $y \neq 0$ is omitted; on page 41, in the statement of the lemma, $u \neq 0$, and in § 57, $k \neq 0$. In light of the complexity of the subscript notation needed, it is surprising that there are so few typographical errors. As the author says, in closing the introduction, "the quality of the printing speaks for itself." There is no index, but the table of contents is so detailed that it practically takes the place of one.

We have already pointed out the lack of general comment; in this connection we wish now to note the meagreness of the treatment accorded to nilfactors. We should like to have seen such an important concept introduced at the first opportunity, namely in § 4, by calling attention to the fact that the algebras given as examples differ essentially, in that the matric algebra contains nilfactors whereas $F(i)$ does not. And it would be natural for us to introduce the definition of nilfactors in § 6 in connection with $\Delta(x)$ and $\Delta'(x)$, and point out that they are simply the numbers for which these determinants vanish. As a matter of fact, nilfactors are not defined until § 19 and then only for an associative algebra with a principal unit ϵ, whereas the concept may be extended to any algebra; we learn there that $x - \omega\epsilon$ is a nilfactor if and only if ω is a root of the rank equation. This is the first and last mention of "nilfactor" in Parts I, II. It is not even pointed out that the nilpotent number is a special type of nilfactor. We learn that a necessary and sufficient condition that a number be nilpotent is that every root of its characteristic equation be zero, but nowhere are we told that the corresponding condition for a nilfactor is that one root of its characteristic equation vanish.

The conditions on the γ_{ijk}, that multiplication be associative, all but escape mention; we find them buried in a proof on page 59. If the author was to accord them a place at all—as he certainly should—should it not be their natural one in § 5, where multiplication is defined and discussed? Finally, the following points did escape mention: an associative algebra with n units with (without) a principal unit, which is of rank $n(n+1)$, is commutative; a binary algebra with a principal unit is associative and commutative (essential to the complete understanding of § 18).

Though we have had differences with the author, we are in the main at one with him. He has written an able and comprehensive book on an abstruse subject, and a book which satisfies a long felt need, as an introduction and also as an up-to-date book of reference. It should prove invaluable to the beginner and advanced student alike.

W. C. GRAUSTEIN.

SHORTER NOTICES.

Wissenschaft und Methode. Par H. POINCARÉ. Autorisierte Deutsche Ausgabe mit erläuternden Anmerkungen von F. und L. Lindemann. Leipzig und Berlin, Teubner, 1914. vi+283 pp. 5 Marks.

THIS is a straightforward translation of the Science et Méthode of Poincaré, and although the original has never been reviewed in the BULLETIN, it is scarcely necessary at this late date to do so, since every one is familiar with it. In case one were not, he needs only to be informed that the book is the third in the famous series of Poincaré on Science and Hypothesis, the Value of Science, Science and Method, and Last Thoughts. The chief principle of this work is the substantial identity of the methods of science and those of mathematics. The remarkable address on mathematical invention occurs here, in which Poincaré states his views of how mathematical creation takes place, illustrating his remarks from his own experience. In brief the explanation is that after a long and severe study of the nature of a problem, one need not keep on hammering away indefinitely, but should remember that if the right idea for the solution ever comes, it will flash out suddenly and at even inopportune moments, often many days, or weeks, or even years, after the first consideration of the problem. He explains the attractiveness of the right idea when it comes before the mind, by ascribing to our esthetic sense the power of arresting and holding such ideas as fit in with the harmony of the intellectual world we build. It then becomes the function of our logical faculty to verify formally the truth of what has presented itself to us in this way.

The same general notions underlie his statement of the method of the scientist whether in the selection of his facts or in the general development of his theories. The first part of the book discusses from this standpoint the scientist and his work. The second part discusses mathematical educational method. The third part discusses the mechanics of the relativity theory. The fourth part consists of two chapters: in one, a study of the Milky Way by the kinetic theory of gases; in the other, a description of the precise methods of a geodetic survey. Each of these parts contains the well-known ideas of the author in various places. Each part has the stimulating character of all of the philosophical work of the eminent mathematician, physicist and philosopher. It is a rare occurrence for one so profound in each of these lines to be also so skilled in presenting as clear an exposition of the working of the intellect. The best criticism we can make on the book is that no one can afford to be without it or the original.

JAMES BYRNIE SHAW.

A Text Book of Mechanics. By LOUIS A. MARTIN, JR. Vol. 5, *Hydraulics.* New York, John Wiley and Sons, 1914. 12 mo. 223 pp.

THIS is the concluding volume of a series, the earlier volumes of which (Statics, Kinetics, Mechanics of Materials, Applied Statics) have been previously reviewed in this BULLETIN.*

Approximately 50 pages are devoted to the discussion of liquids at rest; the force exerted by a liquid upon a plane or curved surface; methods of measuring pressure and locating the centre of pressure; applications to gates, locks and dams; floating bodies and the conditions for stability. The remainder of the book treats of liquids in motion, starting with an elementary discussion of the free surface of liquids moving with acceleration. Then comes Bernouilli's classic theorem on the flow of liquids, followed by a treatment of flow through orifices, various formulas for weirs of several types, flow through pipes with applications to hydraulic transmission of energy and calculation of losses in pipe lines, force exerted by a moving liquid upon pipes and on stationary deflecting surfaces, axial or radial flow, impulse wheels and turbines, and their design. There is a good set of problems for a final review.

As in the earlier volumes, the exercises distributed through the text play a vital part in the course, giving students a chance to work out parts of the subject for themselves. These exercises are not too difficult, having been introduced so carefully as to fall within the power of average students. We should add that throughout the series of texts the author presupposes practically no knowledge of physics, stating carefully the basic principles which are to be employed.

The mathematical equipment assumed, however, steadily increases through the series, so that the first volume may be studied before the first course in calculus, while this last volume assumes a very fair knowledge of integral calculus.

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* Vol. 16 (1909), pp. 144–7; vol. 21 (1914), pp. 140–3.

The present volume, like those which preceded it, gives every indication of representing an actual course, skillfully taught. The series of volumes afford a most satisfactory introductory course in theoretical and applied mechanics.

F. L. GRIFFIN.

NOTES.

THE closing (June) number of volume 16 of the *Annals of Mathematics* contains the following papers: "Projective classification of cubic surfaces modulo 2," by L. E. DICKSON; "Normal forms for one- and two-sided surfaces," by J. W. ALEXANDER, II; "Minima of double integrals with respect to one-sided variations," by C. A. FISCHER; "A new proof of Sylow's theorem," by G. A. MILLER; "Non-linear integral equations of the Volterra type," by H. GALAJIKIAN; "On some general theorems concerning ordinary closed curves," by ARNOLD EMCH.

PROFESSOR ARTHUR RANUM, of Cornell University, has been appointed an associate editor of the *Transactions of the American Mathematical Society*.

AT the meeting of the London mathematical society held on April 22 the following papers were read: "Note on Dirichlet's division problem," by G. H. HARDY; "Note on a new form of closed linkage," by R. L. HIPPISLEY; "Division of the lemniscate into seven equal parts," by G. B. MATTHEWS; "The influence of the oceanic waters on the law of variation of latitude," by J. LARMOR.

AT the meeting of the Edinburgh mathematical society on May 14, the following papers were read: "On the roots of a derivative of a rational function," by L. R. FORD; "Study of the life and writings of Colin Maclaurin," by C. TWEEDIE; "Recurrence formulas for the functions which represent solutions of the differential equation $d^2u/dx^2 - a^2u = p(p + 1)u/x^2$," by H. T. FLINT; "Two simple nomograms," by E. T. WHITTAKER.

THE eighty-fifth annual meeting of the British association for the advancement of science will be held at Manchester, September 7–11, under the presidency of Professor A. SCHUSTER; Sir F. D. DYSON is chairman of section A, mathematical and physical sciences.

THE following courses in mathematics are announced for the present summer semester:

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ: Integral calculus, with exercises, six hours; Applications of elliptic functions, four hours; Elementary geometric treatment of certain problems in maxima and minima, one hour; Colloquium, two hours; Seminar, two hours.—By Professor G. FROBENIUS: Analytic geometry, four hours; Seminar, two hours.—By Professor F. SCHOTTKY: Theory of ellliptic functions, four hours; Plane differential geometry, two hours; Seminar, two hours.—By Professor J. KNOBLAUCH: Partial differential equations, four hours; Theory and application of determinants, four hours; Selected chapters from the theory of surfaces, one hour.—By Dr. A. BYK: Non-euclidean geometry and mechanics, one hour.—By Dr. B. GEHRCKE: Elements of higher mathematics, two hours.—By Dr. C. KNOPP: Ordinary differential equations, four hours; Theory of functions, four hours.—By Dr. D. ROTHE: Differential calculus, with exercises, five hours.

UNIVERSITY OF BONN.—By Professor E. STUDY: Differential geometry, four hours; Seminar, two hours.—By Professor F. LONDON: Elements of differential and integral calculus, four hours; Axonometry and perspective, two hours.—By Professor I. SCHUR: Algebraic equations, four hours; Theory of sets, two hours; Seminar, two hours.—By Dr. J. O. MÜLLER: Selected chapters of integral calculus, one hour; Introduction to differential equations, three hours; Seminar, two hours.

UNIVERSITY OF FRANKFORT.—By Professor A. SCHOENFLIES: Differential calculus, with exercises, five hours.—By Professor L. BIEBERBACH: Algebra, four hours; Algebraic curves, two hours; Seminar, two hours.—By Professor E. HELLINGER: Theory of functions, five hours; Seminar, two hours.—By Dr. O. SZÁSZ: Analytic geometry, with exercises, five hours.—By Professor E. BRENDEL: Spherical astronomy, two hours; Mathematics of insurance, with exercises, four hours.

UNIVERSITY OF GÖTTINGEN.—By Professor D. HILBERT: Calculus of variations, four hours; Selected chapters on the structure of the plane, two hours; Seminar, two hours.—By Professor E. LANDAU: Applications of the calculus and the theory of functions to the geometry of numbers, six hours; Seminar, two hours.—By Professor C. CARATHÉODORY: Theory of functions, four hours; Partial differential equations of the first order, two hours; Seminar, two hours.—By Professor C. RUNGE: Calculus, with exercises, six hours; Seminar, two hours.—By Professor L. PRANDTL: Elementary mechanics, four hours; Seminar, two hours.—By Dr. H. v. SANDEN: Numerical calculations, with exercises, six hours.—By Dr. E. HECKE: Analytic geometry, with exercises, five hours.—By Dr. L. COURANT: Partial differential equations of mathematical physics, four hours; Selected chapters of the theory of functions, two hours.—By Dr. F. BERNSTEIN: Analytic foundation of the theory of probability, two hours; Mathematics of insurance, two hours; Seminar, two hours.

UNIVERSITY OF LEIPZIG.—By Professor O. HÖLDER: Higher algebra, four hours; Applications of the theory of elliptic functions, three hours; Seminar, two hours.—By Professor K. ROHN: Plane analytic geometry, with exercises, five hours; Descriptive geometry, with exercises, four hours; Determinants, two hours.—By Professor G. HERGLOTZ: Theory of numbers, four hours; Definite integrals and Fourier series, two hours; Seminar, two hours.—By Dr. K. BLASCHKE: Ordinary differential equations, with exercises, six hours; Algebraïc analysis, two hours.

UNIVERSITY OF MUNICH.—By Professor F. LINDEMANN: Analytic geometry of space, five hours; Theory of elliptic functions, five hours; Seminar, two hours.—By Professor A. VOSS: Analytic mechanics, four hours; Introduction to hydrodynamics, three hours; Seminar, two hours.—By Professor A. PRINGSHEIM: Higher algebra, II, four hours.—By Professor H. BRUNN: Elements of higher mathematics, three hours.—By Professor G. HARTOGS: Synthetic geometry, four hours.—By Dr. F. BÖHM: Theory of probabilities, four hours; Seminar, two hours.—By Dr. H. DINGLER: Elementary mathematics, four hours; Mathematical treatment of observational data, two hours.—By Dr. A. ROSENTHAL: Differential calculus, with exercises, five hours.

PROFESSORS L. PRANDTL and E. LANDAU, of the University of Göttingen, have been elected to membership in the Göttingen academy of sciences.

THE Helmholtz medal of the Berlin academy of sciences has been awarded to Professor M. PLANCK.

To the list of anniversaries of German professors of mathematics chronicled in the May BULLETIN should be added the celebration by Professor F. MERTENS, of Vienna, of his golden doctor jubilee on November 7, 1914, and his seventy-fifth birthday on March 20, 1915, and by Professor GEORG CANTOR, of the University of Halle, of his seventieth birthday on March 3, 1915.

THE College entrance examination board has now six examiners and twenty-five readers in mathematics, representing twenty-two different schools, colleges, and universities. Professor J. W. YOUNG is chief examiner, and Professor VIRGIL SNYDER chief reader in geometry; Professor F. S. WOODS chief examiner, and Professor C. R. MACINNES chief reader in algebra. Harvard, Princeton, and Yale Universities will discontinue their own entrance examinations after the present year, and will rely entirely on the board examinations.

PROFESSOR A. D. PITCHER, of Dartmouth College, has been appointed professor and head of the department of mathematics in Adelbert College of Western Reserve University. Dr. J. M. STETSON, of the University of Alberta, has been appointed instructor in mathematics at Adelbert College.

PROFESSOR W. M. SMITH, of the University of Oregon, has been appointed associate professor of mathematics at Lafayette College, as successor to the late Professor J. J. HARDY.

PROFESSOR N. F. DAVIS, of Brown University, retires at the close of the present academic year, after more than forty years' active service.

AT the University of Kansas Professor C. A. ASHTON has been promoted from an associate to a full professorship of mathematics, and Professor U. G. MITCHELL from an assistant to an associate professorship of mathematics. Messrs. K. J. HOLZINGER, A. W. LARSEN, and L. L. STEIMLEY have been appointed instructors in mathematics. Mr. H. H. CONWELL has resigned his instructorship to accept an assistant professorship of mathematics in the University of Idaho.

AT Wellesley College Professor HELEN A. MERRILL, at present on leave of absence, has been promoted to a full professorship of mathematics.

Dr. H. W. Reddick, of Columbia University, has been appointed professor and head of the department of mathematics in the Cooper Union, New York City.

Professor R. D. Carmichael, of Indiana University, has been appointed assistant professor of mathematics in the University of Illinois.

By mutual arrangement between the departments concerned, Mr. C. H. Yeaton, recently appointed instructor in mathematics at Dartmouth College, has resigned to accept a similar appointment at Northwestern University, and Mr. C. R. Dines, at present instructor at Northwestern University, has accepted an instructorship in mathematics at Dartmouth College for the coming academic year.

At Cornell University Drs. C. F. Craig and F. W. Owens have been promoted to assistant professorships of mathematics.

At the Massachusetts institute of technology Dr. B. B. Libby and Mr. George Rutledge have been appointed instructors in mathematics.

Dr. G. M. Conwell, of Yale University, has been appointed instructor in mathematics in the New York state college for teachers.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Amoroso (F.) Complementi di analisi algebrica elementare, con appendice sulle sezioni coniche. 2a edizione, riveduta e migliorata. Napoli, Pirro, 1912. 16mo. 34+229 pp.

Brenken (E.). Die Erzeugung der Kurven konstanten Gauss'schen Krummungsmasses auf Flachen zweiten Grades durch elliptische Zylinder. Papenburg, 1914. 17 pp. M. 1.50

Couturat (L.). L'algèbre de la logique. 2e édition. (Collection Scientia.) Paris, Gauthier-Villars, 1914. 8vo. 100 pp. Fr. 2.00

Davison (C.). Subjects for mathematical essays. London, Macmillan, 1914. 8vo. 3s. 6d.

D'Enno (J. G. A.). Ein ganz elementares Verfahren zur Lösung des grossen Fermat'schen Satzes wie auch zur vollständigen Erweiterung und Verallgemeinerung des berühmten Theorems mit Uebungen und Anwendungen auf die Geometrie fur die höheren Klassen der Mittelschulen. Trient, G. B. Monauni, 1912. Gr. 8vo. 23 pp.

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